Geophysical Fluid Dynamics – I P.B. Rhines

Notes on the Boussinesq Approximation - I.   27 ii 2004

The basic vertical momentum balance in geophysical fluids is hydrostatic, if the aspect ratio of the flow is small, $(H/L)^2 << 1$. The vertical forces involved are large, of order $\rho g \sim 10^4$ N m^{-3} for ocean water while the forces involved in fluid motion are of order $\rho f U \sim 10^{-3}$ N m^{-3}. The vast difference, a factor of 10^7, hints at the validity of hydrostatic balance, but of course we want this balance to hold true for the motion-induced pressure p' as well as the basic background stratification and pressure.

Let

$$\rho = \rho_0(z) + \rho'(x,y,z,t)$$
$$p = p_0(z) + p'(x,y,z,t)$$

and the approximate momentum equations become

$$\partial p_0 / \partial z = -g\rho_0$$
$$\rho_0[\partial \vec{u}/\partial t + \vec{u} \bullet \nabla \vec{u} + 2\vec{\Omega} \times \vec{u}] = -\nabla p' - \rho' \vec{g} + \rho_0 \nu \nabla^2 \vec{u}$$

Here we keep the background density, ρ_0, in the inertia of the fluid (lefthand side) and use the motion-induced density ρ' in the buoyancy (having subtracted off the top equation for the basic vertical balance). The simplest set of equations then assumes the fluid incompressible, $D\rho/Dt = 0$. This would give us a buoyancy frequency N, in which the compressibility correction is not there; it is not accurate enough for either atmosphere or ocean.

Let us improve on this by allowing some compressibility of the fluid as it rises or falls. For adiabatic motions (not heat transfer between a fluid parcel and its surroundings)

$$D\rho/Dt = \partial \rho/\partial p|_\eta \; Dp/Dt$$
$$= c^{-2} Dp/Dt$$

where c is the speed of sound, and η the entropy. Now expand the righthand side:

$$D\rho/Dt = c^{-2}(\partial p/\partial t + w\partial p/\partial z + u\partial p/\partial x + v\partial p/\partial y) \qquad (1)$$
$$\cong c^{-2}(\partial p'/\partial t + w\partial p_0/\partial z)$$

where in the last line we keep just the largest term. The ratio of the two last terms is

$$\frac{p'/\tilde{T}}{w\rho_0 g} \qquad (2)$$

where $\tilde{T}$ is the time-scale of the motion. Using the vertical momentum equation

$$\rho_0 Dw/Dt = -\partial p'/\partial z - g\rho' \qquad (3)$$

we see that $p'/w \sim \tilde{T} H/\rho_0$ if the first two terms balance. Then the ratio (2) is

$H/g\tilde{T}^2$.

If the time-scales of interest are those of internal gravity waves and longer, $\tilde{T} > N^{-1}$. The ratio (2) becomes

N^2H/g.

This is useful because the N^2 has a typical size $-g/\rho_0\, d\rho_0/dz \sim g/H_s$ based on the *scale height of the atmosphere or ocean, H_s*. This is defined as the scale over which the basic density gradient would lead to a doubling, say, of the density. A similar result comes from balancing the other two terms in the vertical MOM equation, (3).

Thus a useful set of MASS equations that are more accurate than the incompressible (ρ=constant following a fluid parcel) equations is

$$\nabla \bullet \vec{u} = gw / c^2$$

$$D\rho / Dt = D\rho' / Dt + w\partial\rho_0 / \partial z = -g\rho_0 w / c^2$$

These are called 'anelastic' equations because they cannot describe sound waves, yet they do describe much of the dynamics of GFD. They give the correct value for N^2 which includes the g^2/c^2 reduction in stability. You can compare with the full equations for a compressible fluid in Gill section 6.4. The Boussinesq approximation is much more restrictive and often in text books you will see ρ_0 to be a constant rather than a function of z. The ratio of the two terms in N^2

$$N^2 = -(g / \rho_0)(\partial\rho_0 / \partial z + \rho g / c^2)$$

has a scale-analysis size gH_s/c^2. This is not small for either the ocean or atmosphere. The scale height H_s for an isothermal atmosphere is $RT/g \sim 6.5$ km, whereas from observed T and S profiles the oceanic value of H_s is about 200 km.

To summarize, the anelastic equations allow for compression and expansion of fluid but not sound waves; they are an excellent set to work with. They give the correct potential density and buoyancy frequency. They are accurate locally if $H/H_s << 1$, where H is the height scale of the fluid motion (not necessarily equal to the depth of the fluid layer. The Boussinesq equations most often used in the literature are accurate only if $c^2/gH_s <<1$, which is more restrictive.

A good way to explore these equations is to look at some exact solutions for the exact, complete equations and then see when these approximate equations are valid. Such a solution can be found for internal gravity waves in a stratification of the form

$$N^2 = A \exp(z/d).$$

In a second set of these notes we will explore this. An important result is that, although the Boussinesq approximation may be valid locally, if we have waves of small vertical wavelength ($2\pi H$) that propagate to great altitude in the atmosphere, they will amplify as they encounter lower density. This amplification is a non-Boussinesq effect.