

Review of Ekman layer dynamics for a fluid with uniform density. Gill text, sec. 9.6, then 9.1-9.7

Suppose we have a geostrophic flow in the x-direction, with velocity $(U_0, 0)$. Beneath the flow is a rigid boundary where the velocity is zero, because of viscous effects. Near the horizontal boundary, the velocity is dominantly horizontal; w is small. To start with assume $w=0$, and the horizontal MOM equations become

$$\begin{aligned} -fv &= -\frac{1}{\rho} p_x + \nu u_{zz} \\ fu &= -\frac{1}{\rho} p_y + \nu v_{zz} \end{aligned} \tag{1}$$

where ν is the kinematic viscosity coefficient (= (diffusivity of momentum)/ ρ). Now the interior flow far above the boundary is geostrophic, so

$$fU_0 = -\frac{1}{\rho} p_y$$

and this same pressure gradient penetrates right to the lower boundary, even as though the velocity varies in that layer (because it will be very thin, and the vertical MOM equation then shows that p_z is very small, so the pressure does not vary much in the vertical). So the mathematical problem is to solve (1) with boundary conditions $u+iv=0$ at $z=0$, and $u \Rightarrow U_0, v \Rightarrow 0$ as $z \gg \delta$. The top of the fluid is a free-surface at $z=H$, where we take $w=0$.

Expressing the velocity as $(U_0+u') + iv$, real in the x-direction and imaginary in the y-direction, the equation for the velocity contribution of the boundary layer becomes

$$(u'+iv)_{zz} - i \frac{f}{\nu} (u'+iv) = 0$$

with solution

$$\begin{aligned} u+iv &= U_0(1 - \exp(-z/\delta) \exp(iz/\delta)) \\ u &= U_0(1 - \exp(-z/\delta) \cos(z/\delta)) \\ v &= -U_0 \exp(-z/\delta) \sin(z/\delta) \end{aligned}$$

where $\delta = \sqrt{(2\nu/f)}$. This is a *spiral* velocity pattern, the velocity vector rotating with z as well as changing speed. As one moves downward toward the boundary the speed decreases and hence the Coriolis force is weaker than in the geostrophic flow above. The fluid is then moved in the direction of (the negative of) the pressure gradient, and the viscous force has to arise to balance ∇p .

The *Ekman transport* is the vertical integral of the velocity difference $u'+iv$, (thus it is the extra movement of fluid caused by the Ekman layer. It is

$$M = \int_0^H u' + iv \, dz = \frac{1}{2} U_0 \delta (-1 + i)$$

In particular the volume of fluid moving to the left of the geostrophic velocity vector is $\frac{1}{2} U_0 \delta$. The viscous stress which transmits the force from boundary to fluid is $\tau = (\nu \partial u / \partial z, \nu \partial v / \partial z)$. *Independent of the viscosity coefficient* we find that

$$\vec{\tau} = -\rho \vec{M} \times \vec{f}$$

This stress points 45° to the left of the geostrophic velocity.

In ocean circulation problems we often take the wind-stress as a given boundary condition; in this case the same solution is found for the Ekman layer just beneath the ocean surface; the Ekman volume transport is to the right of the windstress, and it pumps the interior and drives the ocean circulation. Note that since the formula for M is independent of ν , then even with turbulent boundary layers, of realistic thickness, the same effect transport occurs.

We saw this happen in the lab, after we changed the rotation rate of a rotating cylinder, so that the fluid was rotating relative to the floor of the cylinder. Friction wants to bring the fluid to the same speed as the boundaries, so if the initial swirl (azimuthal flow) has cyclonic rotation and low pressure, we need cyclonic deceleration (or anticyclonic acceleration) to slow it down.

This deceleration is caused by Ekman pumping: the vertical velocity, w , driven by the Ekman layer which stretches vortex lines in the geostrophic flow above. w arises when the Ekman flux varies in x - or y -directions. For example if the geostrophic flow varies in y ,

$$\frac{\partial M}{\partial y} = \frac{1}{2} \frac{\partial U_0}{\partial y} \delta$$

If the geostrophic flow points in some other direction, the general result is

$$\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} = \nabla \cdot \vec{M} = -\frac{1}{2} \zeta \delta$$

where ζ is the vertical vorticity of the geostrophic flow. We solved for the flow independent of x and y , and here we make the *boundary layer approximation* that the scale of variation of U_0 is much larger than the scale δ .

Our interior vertical vorticity equation is now

$$\begin{aligned} \frac{D\zeta}{Dt} &= f \frac{\partial w}{\partial z} \quad \text{or} \\ \frac{\partial \zeta}{\partial t} &= f \frac{\partial w}{\partial z} = f (w(z=H) - w(z=0)) / H \end{aligned} \tag{2}$$

the second equation holding if advective rate of change of ζ is small. We can do the final step because with constant density we have the Taylor-Proudman approximation, u and v are independent of z outside the viscous region and w is a linear function of z .

Since $\delta \ll H$, we can solve (2) with the boundary condition

$$w(z=0) = -\frac{1}{2}\zeta\delta$$

This gives us a new equation for the evolution of the geostrophic flow:

$$\frac{D\zeta}{Dt} = -\zeta\left(\frac{1}{2}f\frac{\delta}{H}\right)$$

The solution is

$$\zeta = \zeta_0 \exp(-Rt) \quad R \equiv \frac{1}{2}f\frac{\delta}{H}$$

It is interesting that this spin-down of the flow is what you would have found by supposing that in the MOM equations there were a simple linear drag, $-RU_0$, $-RV_0$. Such an approximation is often made in numerical models of the general circulation. But note that the *viscous overturning circulation* that drives the changes in geostrophic flow (through ζ) carries fluid: we saw purple dye moving in this circulation, up the walls of the cylinder, and back into the interior. None of this would occur with the ‘linear drag law’ model above. Recall we used this linear drag in the wind-driven channel flow early in the term and you might want now to decide what overturning circulation occurs in that problem.

In a constant-density fluid, spin-up and spin-down are achieved by non-viscous vortex stretching driven by viscous boundary layer pumping of the interior.

The time it takes to spin down the flow to rest (in the rotating reference frame) is $1/R$. This is a much faster process than ordinary viscous diffusion. Recall that straight viscous diffusion would spread the boundary effects upward so that $\delta \sim (vt)^{1/2}$. To diffuse all the way through the fluid this process would require a time H^2/ν . The ratio of the two spin-down times is

$$\frac{\text{rotating spindown time}}{\text{nonrotating spindown time}} \sim \frac{2H\nu}{\delta fH^2} = \sqrt{2}E^{1/2}$$

where $E = \nu/fH^2$ is the *Ekman number*; it is small $\ll 1$ in the flows we have looked at and it appears in the scale analysis of the momentum equation carried out in class this week.

A way of remembering this result is that rotating spindown takes a time H/δ times $1/f$; that is [total depth/ 2π x Ekman layer thickness] in ‘days’. In the lab the Ekman thickness δ is about 1 mm ($\nu = 10^{-6}$ m²/sec for water; for air $\nu = 1.5 \times 10^{-5}$ m²/sec).

The spindown time is thus about 200 seconds for $f = 1$ radian/sec whereas the pure viscous diffusion time is about 3 hours (for $H = 0.1$ m).

Understanding general circulation, waves and eddies requires that we understand small-scale mixing, boundary layers and spin-up. Density stratification and buoyancy add much to this problem, but the fundamental results are still partially valid.