GFD Lab Demo V (12 ii 2004) {GFD Lab Demo IV 26 i 2001} **Geostrophic flow, geostrophic adjustment**

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Summary

In this lab we look at the dominant forces acting on fluid on a rotating planet, and their controlling effects on its circulation. This is all best understood by considering the time-dependent adjustment of the fluid flow from simple initial conditions.

Geostrophic flow has a close balance between horizontal pressure gradient force and Coriolis force. It arises for slow, slowly varying flows of large scale on a rotating planet. While in general, motions both rapidly varying and slowly varying, fast and slow, small- and large-scale occur in atmosphere and ocean, there is a 'low-pass filtered' dynamics that applies to large-scale motions seen on a weather map. These motions tend to have a partially 2-dimensional nature: they do not move freely in the vertical (or, normal to surfaces of constant potential density). Geostrophic horizontal momentum balance and hydrostatic vertical momentum balance defines this subclass of motions, but further analysis (often involving the vorticity equation) is needed to write down the full set of equations that describe the time-development of these flows. We step forward in time, first looking at the establishment of geostrophic flow from an arbitrary 'unbalanced' set of initial conditions. Incidentally, a region of active research is the interaction between the fast, rapidly varying non-geostrophic motions (internal waves, frontal jets, surface waves) and geostrophic motions...though we emphasize the evolution toward geostrophic motion, it is actually a two-way street.

Pressure gauge. Pressure is the scalar field whose gradient describes the major large-scale force exerted by a fluid parcel on its neighbors. Large scale motions with 'thin' aspect ratio, $(H/L)^2 << 1$, are hydrostatic, giving us the 'free-surface pressure gauge', showing the pressure just beneath the free surface of a liquid. Of course, if the atmospheric pressure is not uniform, there is an 'inverted barometer effect' which produces variations in the surface elevation. In the interior of a stratified atmosphere or ocean, hydrostatic pressure relates the vertical displacement of isopycnal (constant density) surfaces to vertical pressure gradient. In some useful cases, the topography of these surfaces give us a direct picture of the pressure. Suppose the fluid has just two density layers, ρ_1 and $\rho_{2.}$ The hydrostatic pressure difference between to levels, z_1 and z_z, on either side of the density interface, is just $g(\rho_2 - \rho_2)\eta_{interface}$ plus a constant, where $\eta_{interface}$ is the interface elevation. The surface elevation η and $\eta_{interface}$ together describe two modes of the fluid; these were visible in the Waves-I lab. If the lower layer is very much thicker than the upper, there is an interesting mode in which the energy is concentrated in the thin upper layer. This is called a '1 ½ layer' model. Then $\eta_{interface}$ is proportional to the upper layer pressure, itself.

(i) Geostrophic adjustment in a single layer of fluid with a free surface. In lab III we saw that Coriolis forces endow a fluid with a sort of elasticity, giving rise to oscillations, waves and unexpected force balances. They also modify familiar non-rotating phenomena like long gravity waves.

If a single layer of fluid, of mean depth H, is disturbed by lifting the surface and then releasing it, gravity waves carry the adjustment signal outward, leveling the free surface. Suppose that the initial form of the surface is a Gaussian hill, of width L. In absence of Coriolis forces the hill breaks into two smaller hills, one propagating left and one right. Supposing H/L <<1, the waves are long, non-dispersive, with speed $c_0 = \sqrt{(gH)}$. Now the fluid itself moves in the direction of propagation, beneath each hill, and in doing so removes the mass and potential energy excess of the initial conditions, to infinity.

If the fluid is rotating, these velocity vectors initially veer to their right (in the northern hemisphere) with angular velocity exactly f^{-1}. Supposing the wave pulse passes in a time much less than this the Coriolis effect is slight; this time of passage is roughly L/c_0. Yet if it takes a significant fraction of the Coriolis period to pass, the wave pulse, and indeed its escape from the initial location, is altered. Long-channel (y-direction, v-velocity) flow develops. The ratio of these times, say γ, is

$$\gamma = fL/c_0 \equiv L/\lambda$$

where $\lambda = c_0/f$ is the Rossby deformation radius. γ^2 is known as the Burger number, and it has wide application in systems where buoyancy forces and Coriolis forces are in competition. This definition of λ is quite good, applying much more generally to internal modes of motion in a continuously stratified fluid, where each mode has its own phase speed and its own λ.

After the waves have departed, the remaining hill of fluid comes quickly (in time ~ f^{-1}) to geostrophic balance. In the 1-D channel this balance involves a pair of jets into and out of the page, for geostrophy is a relation between pressure *gradient* and velocity, hence surface slope and velocity. We can instead do a ‘one-sided’ experiment in which the initial surface profile is a step discontinuity at x=0. This is Rossby’s original calculation). Now, the wave activity tries to move fluid to from the deep side to the shallow side of the discontinuity, to level the free surface. A single jet develops from Coriolis forces, into the page. In Gill’s (1982) presentation, the mathematical wave equation turns out to be impossible to solve on its own, until a constraint from the potential vorticity principle is brought into play.

In fact, the effects of Coriolis can be restated in terms of potential vorticity, $q = (f+\zeta)/h$: free surface slumping, with columns of fluid squeezing to smaller height, requires that some relative vorticity, ζ, appear in exchange for planetary vorticity, f. If f is positive, then ζ will be negative, that is anticyclonic, just as we have seen from momentum arguments.

The numerical simulation shows how, with increasing Coriolis frequency (or increasing lateral scale, or decreasing g), potential energy, PE, is increasingly trapped at its point of injection and prevented from converting to kinetic energy, KE. In fact γ^2 also expresses the ratio PE/KE, for

$$KE = \iiint_V \tfrac{1}{2}\rho |\vec{u}|^2 \, dV, PE = \iiint_V \rho\Phi \, dV; \Phi = gz$$

If the fluid surface lies at z=η(x,t), for a single-layer fluid $PE = \tfrac{1}{2}\rho g \iint_A \eta^2 \, dA$. Use a geostrophic estimate of the horizontal velocity, U ~ gη/fL. Then the ratio PE/KE is, using scale analysis, $g\eta^2/HU^2 \sim f^2L^2/gh \equiv \gamma^2$.

In the laboratory, and in many natural flows, the full force of gravity is not operating, but we set up density-layered fluids in which the height of a density interface rather than the fluid surface is the active variable. A particularly useful idealization is called the 1 ½ layer model: a very deep layer of density ρ_0 beneath a much thinner layer of density $\rho_0 + \delta\rho$. In this case, somewhat surprisingly, the deep lower layer is dynamically passive, and the wave mode of interest has its energy concentrated in the thin upper layer. In an idealized sense, the model behaves just like a one-layer model, but for a reduction of gravity to a new value $g' \equiv g\delta\rho/\rho$.

What the parameter γ tells is how near the initial conditions are to a state of geostrophic balance: near if γ is small, far if γ is large. The initial conditions tend to project on the geostrophic solution closely, when γ is large. This depends to some extent on flow configuration.

Infinite channels are a useful idealization, but for the purposes of laboratory experiments, and many applications in Nature, we wrap the channel into a circle, and look at cylindrically symmetric adjustment. Now we have a circular hill of fluid, with zero velocity in the rotating frame, released at time t=0. The physics is much the same, only that the outward propagating waves, with circular wave crests, diminish in amplitude due to simple geometric spreading of their rays.

The gravitational collapse of the unbalanced hill of fluid forces a radially outward flow. Coriolis forces turn this velocity to the right, and the dome begins to spin anti-cyclonically. Spin builds up until the new Coriolis force to *its* right (that is, radially inward) begins to balance the outward pressure force. The visual connection between shrinking thickness of the water mass and increasing anticyclonic vorticity, is strong.

Apparatus:

- rotating table
- container (glass, plexiglass cylinder or kitchen bowl)
- salt
- food coloring
- eye-dropper

Let a bowl nearly filled with salty water come up to speed on the turntable (say, 50g of salt per liter of water, which gives a density of about 5% greater than fresh water). A speed of about 1 radian sec^{-1} is fine. Now take fresh water with some dye in it, and drip it on to the center of the water surface…carefully so as not to cause too much mixing. The complex blob of buoyant water will gather together into a more circular mass, and begin to spin. Its spin may be hard to see (without riding round with the rotation), but if you lay down radial dye lines you will see the spin imprinted on them. The spinning mass takes on a lenticular shape, as seen from the side. You can estimate γ from the radius of the colored lens of buoyant water, the rotation rate Ω, and the density difference $\delta\rho/\rho$, as expressed in the reduced gravity g'.

You may well see the circular symmetry break down, and small eddies form at the rim of the colored lens. This important instability will be developed in later labs.

If γ is large, the rim of the spinning lens is very like the 1-dimensional Rossby adjustment problem above, with an anti-cyclonic jet concentrated there.

(ii) Rossby-Gill adjustment in a channel and Kelvin waves *This experiment was not part of the 2004 lab but is part of our discussions.*

In looking at long waves without rotation we finished off the experiment by pouring a beaker of dense salt water into the wave tank, and watched it surge back and forth as a gravity current plus solitary waves: another form of undular bore. What if the channel were rotating?

Apparatus:

- rotating table
- channel shaped container (in our case 120cm x 30 cm x 30 cm)
- ‘dam’
- salt (~250g. made up to roughly 50 l. of 0.5% salinity fluid)
- food coloring

eye-dropper
Boris' stratification raft

Our apparatus for this experiment has evolved over the years. We want to introduce an initial condition of elevated free surface or depressed density interface, with no initial velocity. A very effective way to do this is to construct a cubical box that just fits inside the channel. The cube is open at top and bottom, and stands on little legs so that fluid is free to communicate near the bottom. The cube can be of any material, but plexiglas itself makes for good visibility.

The channel filled to at least 2/3 of its height with salty water (roughly 20 cm). It is centered on the rotating table (which may be a small table top apparatus, if is strong enough…a potter's wheel is particularly good. Remotored phonograph turntables may work, although an *aboriginal* record player is probably of insufficient power). The table is set rotating: a variety of speeds will be of interest but start slowly: as slow as 60 sec per revolution. When the fluid has reached solid-body rotation color 7 l. of fresh water with green dye and 1 l. with red dye. Slowly pour the larger container of fresh water (which will add about 2 cm. depth) onto the free surface using the Boris raft to forestall mixing.

When this two-layer system has settled down insert the cubical dam at one end of the channel. Add another cm. of red salty water using the raft, and then remove the raft. When all the fluid is once again motionless note the difference in interface and free surface elevations within and without the cube. Now carefully lift the cube out of the channel.

Geostrophic adjustment occurs along the boundary between salty and fresh water, with the dyed fluid moving down the channel, being deflected to the right by Coriolis forces, and quickly (a time ~ f^{-1} ~ 1/12 of a revolution of the table!) coming close to geostrophic balance in which a jet of flow runs along the boundary between the water masses.

But, unlike the circular flow in (i) walls deflect the boundary jet, which runs down the channel. This boundary current is a new aspect of geostrophic flow: by leaning against the wall, the jet's impetus to turn right is blocked. There is a close connection between this flow and the *Kelvin wave*, which is a gravity wave trapped by rotation (Gill, 1982). It is worth doing more realizations of the above experiment, to focus on the boundary wave propagation. A recurring theme in these notes is the role of waves in establishing mean flows. Here the Kelvin wave races down the wall of the channel ahead of the dyed fluid. It sets up the boundary current, and the dyed fluid follows this preset path. The appropriate long-wave speed for the Kelvin wave is $c_0 = \sqrt{g'H_1}$ ~ 4.5 cm sec^{-1} where H_1 is the depth of the colored upper layer. By contrast, the dyed fluid follows down the channel with velocity ~ εc_0 where ε

is the measure of amplitude that we gave in discussing waves: $\varepsilon = \eta/H$ where η is the thickness difference in the upper layer.

The Kelvin wave/boundary currents have a width very roughly given by $\lambda = c_0/f$, the Rossby radius of deformation. At rotation rates of 0.5 radian sec^{-1} or so , this will be quite narrow, and if dye stripes are laid down before the experiment begins, you can see the wavefront advance down the boundary (via its velocity signal) well before the dyed fluid has moved along the same path.

Linear and nonlinear. The amount of colored fluid, the extra fluid added to the upper layer which creates the potential energy for the flow, decides the amplitude of this flow. With a slight volume of new fluid the fluid velocity is much less than c_0 and Gill's solution holds. However, consider the *potential vorticity* of this flow: the dyed fresh water in the cube is thicker (deeper) than the dyed fresh water outside. As it adjusts, therefore, the vertical columns of 'red' fluid will decrease in height. As soon as this fluid flows along the interior jet and turns along the boundary, it will have negative relative vorticity, which is just wrong for the boundary current seen in the experiment! This flow has quite an elaborate life cycle in fact: first, Rossby adjustment along the boundary between the water masses; then, Kelvin wave forerunners shooting round the boundary; then, as fluid moves along this circuit, potential vorticity conservation dictates a change in shape of the boundary current velocity profile and, in fact, the negative vorticity causes a slow extension of fluid down the opposite wall, in the 'anti-Kelvin wave' direction. It is a miniature lesson in the progress of science: to Rossby's 1939 result (stage one) were added Gill's boundary jets (1982), followed by the nonlinear vorticity stage (Hermann, Rhines and Johnson 1988). However, as you see by doing the experiment, these 3 stages are followed by a 4^{th}, which is beyond simple calculation: the boundary jets become unstable and the channel is filled with 'roundish' eddies. In all the miniscule PE that drives the flow will not be exhausted for many tens of minutes, or perhaps several hours.

The experiment can be repeated without introducing the ~2cm. layer of dyed upper layer fluid. Then, a buoyant colored layer is simply added on top of a single layer of denser water. This is the very-nonlinear limit in which the advancing Kelvin wave is replaced by a rotating 'gravity current'. The result is quite similar, but the strong theoretical underpinning, and the presence of rapid forerunner waves, is totally lost.

In similar fashion, one can see the Kelvin gravity current by simply pouring some salt water into a rotating container of freshwater. The dense water finds the nearest wall, and runs along it in the direction of a Kelvin wave. A variant on this is the

River outflow plume. In the channel apparatus, with a deep lower layer of saline water, inject a steady source of fresh water at the surface, by siphoning it from a beaker. Dye the fresh water. Once again the river 'turns right' and flows round until circumnavigating the channel. But there is another feature: usually some of the water will pool in an anticyclonic eddy that sits near the 'river mouth'. This is the same potential vorticity event that we discuss above, where a fluid layer that slumps under gravity will become thinner, and conserving its potential vorticity for a time, will spin anticyclonically. If the river flow is altered or pulsed regularly, the anticyclone may propagate off by itself in the 'anti-Kelvin wave' direction (to the left). Apparently, the tendency for the anticyclonic vorticity to propagate with its image cyclone, as a dipole, is normally opposed by the rightward advection of the continuous river plume. When this is stopped, the eddy is freer to propagate. The idea of the inertial radius, U/f, is useful in thinking about the curvature of the river plume.

Vortex dipoles. This experiment is valuable because it illustrates two fundamental kinds of boundary interaction with a flow: first, with boundary waves and ensuing jets and second, with a region of vorticity seeing its image in the wall, and hence advecting as a vortex dipole along the boundary.

(iii) Thermal wind shear: the fundamental form of winds and currents. When a cold front arrives in the atmosphere, it is accompanied by strong winds, which rotate (veer or back) rapidly as the front passes, and then tend to subside. The sloping frontal surface corresponds with strong variation of the horizontal velocity with height. More widely, the horizontal variation of temperature seen on a weather chart corresponds closely to the *vertical* variation in wind. The experiment described here gives a quantitative look at this remarkable phenomenon, in a controlled manner.

The outcome of the previous experiments is a flow that has hydrostatic ('pressure = weight of fluid overhead') vertical force balance, and geostrophic ('pressure gradient balances Coriolis force') horizontal force balance. These account for the dominant fraction of large-scale motion of the oceans and atmosphere, typically for motions whose *Rossby numbers* $Ro \equiv U/\Omega L$ and $\varepsilon = 1/\Omega T$ are both small, and whose *aspect ratio* H/L is also small. This typically means flows with length scale greater than a few km. and time scale greater than 4 hours. We saw in Lab 3. how ordinary water, when rotated aquires a kind of 'stiffness' along the z-axis (parallel with □). This property allows both inertial waves and, at longer time scale, geostrophic flows with the Taylor-Proudman property, where horizontal velocity is independent of z.

Now we come to the essence of the density-stratified, rotating fluid. The Taylor-Proudman effect due to rotation is seriously confronted by buoyancy forces, which suddenly allow a vertical variation in horizontal velocity (a 'shear'). Indeed, everyone knows that winds in the atmosphere and currents in the ocean vary with z, and there must be a dynamical explanation. This is best understood in terms of the horizontal vorticity balance, where the twisting (that is, the curl of a vector force) due to the tilted surfaces of constant density comes to balance a tipping over of the planetary (Ω-related) vorticity by the shearing flow. An analogy can be found in the precession of a gyroscope by a gravitational torque, but surprisingly few students these days seem to have heard of the gyroscope! Perhaps for the present you might just note that the low-frequency inertial waves which hold the 'fabric' of the Taylor-Proudman fluid together, are no longer present with significant density stratification.

Apparatus:

- rotating table
- clear right-circular cylinder (of plexiglas or glass) roughly 25 cm diameter
- smaller cylinder (of any material) that leaves a gap of roughly 3 cm.
- thermometer (glass or electronic, with long probe tip)
- piece of wood drilled to accept the thermometer (see fig. xx)
- ice
- dye
- long-nosed eye-dropper

Place the smaller cylinder, centered, within the larger cylinder. Small dots of modelling clay help to hold its position. Place some scraps of metal inside it, so that it will not float away. Fill it with crushed ice or ice cubes and water. Now fill the gap between inner and outer cylinders, center them on the table, and rotate at about 1 radian sec^{-1} (12 sec. rotation period). If using a standard r.p.m. phonograph (2 sec. rotation period), the flow may be a little less simple than ideal, but it still will suffice.

After a few minutes, make a vertical dye stripe with a long dropper. Assuming the fluid has come up to speed with the table, you should see the dye line tip over systematically with time. Our proposition is that the rate of tipping over is proportional to the horizontal density gradient (hence horizontal temperature gradient) in the fluid. To test this, insert the thermometer in its wooden holder, so that the sensing element measures the temperature close to (not touching) the inner wall. Then repeat, measuring temperature close to the outer wall. Measure the time it takes for a vertical dye streak to tip over, to the point that its top has made one complete revolution of the annulus, relative to its bottom. This gives an average value for $\partial v/\partial z$. The laboratory thermal wind equation is (for one scalar component, v)

$$\frac{g}{\rho}\frac{\partial\rho}{\partial x} = -2\Omega\frac{\partial v}{\partial z}$$

$$g\alpha\frac{\partial T}{\partial x} = 2\Omega\frac{\partial v}{\partial z}$$

where $\alpha \equiv -(1/\rho)\partial\rho/\partial T$ is the thermal expansion coefficient for water (see appendix in Gill's text). You may find that the shear is stronger near top and bottom than in the middle, in which case a correction may be made. In our experience the equation balances to within about 25% without too much attention to such details.

What is the logic behind this? Consider the simplest geostrophic adjustment: uniform-density fluid in a channel has a tilted upper surface, yet no velocity. When released, the surface levels out due to gravity, and a cross-channel flow occurs. This flow is deflected down the channel (into the page) and, finally it feels a Coriolis force directly opposing the pressure gradient that created it. Now replace the free upper surface with a second layer of buoyant fluid. The interface between the two layers is initially tilted. As the denser, lower layer fluid flows to the left, driven by the hydrostatic pressure gradient, the less dense upper layer flows to the right, to replace it. Now, the down-channel velocities develop in the two layers *in opposite directions.* This is the essence of thermal-wind shear.

To complete the story, we finally look at a continuously stratified fluid, in which there is a family of surfaces of constant density. Using the same tilted initial condition, the velocity across the channel forms a 'gyre' of leftward, upward, rightward and downward flow. Once again down-channel Coriolis force on this flow accelerates a pattern of vertically sheared horizontal velocity, which comes into balance with the residual tilt of the constant-density lines. Note the extent to which the experiment fits this latter picture of roughly symmetrical inflow and outflow, and roughly straight-line velocity profile.

This experiment has the important effect of testing our quantitative skill, which is comparatively rare in this series of mainly qualitative demonstrations. It is good at this point to consult a typical weather map and make the same test. For a perfect gas approximation to air, $\alpha = -(1/\rho)(\partial\rho/\partial T) = -1/T$, with T measured in degrees Kelvin. Thus, at 20^0C or 293^0K, one scalar component of the thermal wind equation becomes .

$$\frac{\partial v}{\partial z} = 3340\frac{\partial T}{\partial x}$$

In air, a 10 degree temperature change in 1000 km horizontal distance thus balances a 33 meter sec^{-1} vertical change in horizontal wind, over 1 km height difference. In terms of density ρ,

$$f\,\frac{\partial \vec{u}}{\partial z} = \frac{\vec{g}}{\rho} \times \nabla \rho; \qquad f \equiv 2\Omega\ \sin\ (\textit{mean latitude}), \vec{g} = -g\vec{\hat{z}}$$

where is a vertical unit vector. For the ocean, with $\nabla\rho = -\rho\alpha\nabla T + \beta\ \nabla S$ (see appendix in Gill's text), so that one componet is just

$$\frac{\partial v}{\partial z} = 19.2\frac{\partial T}{\partial x} - 0.76 \times 10^4\,\frac{\partial S}{\partial x}$$

in sec^{-1} (that is, meters per second per meter of z), for midlatitude, $T=13^0C$, $f = 10^{-4}\ sec^{-1}$, $\alpha = 1.96 \times 10^{-4}$, $g = 9.8\ m\ sec^{-2}$. S is given in parts per thousand, or 10^{-3} kg (salt)/(kg seawater) In the Gulf Stream, a 10^0C change in temperature over 50 km gives $\partial v/\partial z \sim 4 \times 10^{-3}$, or 1 meter sec^{-1} velocity change over 250m vertical distance.

Visualize the *isotherms* in this fluid. These constant-temperature surfaces tilt upward/inward, expressing the pair of facts: it is colder on the inner wall, and it is colder at the bottom than the top. Use the thermometer to probe a few different depths and sketch the isotherms. If it can be done, a vertical profile of temperature, though it does not appear in the formula, is nevertheless useful to see whether cold water 'pools' at the bottom.

As the rotation rate or the heat flux due to the ice is changed (for example, by insulating the inner cylinder, or using a metal coffee can instead of a plastic cylinder) the thermal wind will change, with rapid rotation or small temperature gradient both corresponding to small velocity.
More essentially, with rapid rotation or small density gradient you will find the circular flow going unstable to a beautiful pattern of eddies and jets. But, this is the subject for later work.

(iv) Stratified geostrophic adjustment in a cylinder. This conclusion of the series of experiments adds a number of surprising features, and illustrates a kind of circular eddy motion that is common to both ocean and atmosphere. The object here is to create a *baroclinic vortex* through geostrophic adjustment.

Apparatus:

rotating table
clear-walled cylinder (as large as possible)
funnel and fitted glass or plastic tube
3% salt water (½ volume of cylinder)
1.5% salt water dyed

A two-layer fluid is created by half filling the cylinder with fresh water, and then introducing lower, 3% saline layer using a siphon and diffuser. The fluid is then spun up on the rotating table. In our laboratory, using a 50 cm diameter, 50 cm height cylinder spin up requires typically 2 hours. The extension tube from the funnel is now inserted in the fluid at its center. About 25 cc of 1.5% salinity water, dyed for visibility, is poured in. It may be desirable to cover the end of the tube with a cloth mesh so as to slow the fluid entering.

The introduced fluid has no angular momentum, so that as it spreads radially outward, it develops a strong anticyclonic swirl velocity (in the rotating frame). This is just another way of saying that a Coriolis force is exerted perpendicular to the radial velocity. This swirl, in turn, has an inward Coriolis force that eventually balances the outward pressure force. A lens-shaped, rotating water mass is formed, which usually develops nearly perfect circular symmetry. In our laboratory, a video camera looks down on the table, and rotates with it. A time-lapse video recorder gives us an 'instant replay' of the experiment in accelerated time.

Eddies like this are found in the oceans, particularly at sites where a strong front exists that can spawn eddies through instability. They are often anticyclonic suggesting a 'collapse' of height of the initial water column. They sometimes live for years, and have been tracked with remarkable deep RAFOS floats.

After the eddy has been observed for awhile, it will probably wander away from the center of the cylinder. Then, inject a second one (of a different color). We now enter a new realm of eddy interactions, a forerunner of later labs. A characteristic of this process is that two eddies of like sign will tend to rotate round one another and coalesce into a single eddy. Of course, a pair of 'point' vortices simply orbit round each other. Here, the finite size of the eddies causes them to deform one another in addition, and this deformation often leads to coalescence.

These experiments give useful intuition for exploring atlases of ocean density (temperature and salinity) or atmospheric temperature cross sections. Wherever tilted isotherms or iso-density lines are seen, there is a vertical velocity shear. Dominant flows like the jet stream and Gulf Stream can thus be diagnosed, to sketch the velocity itself (subject to a possibly unknown integration constant). One caveat: it is *potential temperature* θ and not measured temperature T that is related to the simple density fields we have been discussing, and surfaces of constant θ tip in the opposite direction to surfaces of constant T, in the atmosphere. This is because T tends to decrease upward in the lower atmosphere, owing to the strong decrease in hydrostatic pressure, whereas θ increases upward in the actual atmosphere.

The energy for these long-lived motions is, once again, supplied by the initial PE of the stratification. It is remarkable that so little energy can run the ‘density engine’ for so long. In the last experiment, the fluid may still be circulating after several hours. Our earlier evaluation of the potential energy field is modified as follows:

$$KE = \iiint_V \tfrac{1}{2}\rho\,|\vec{u}\,|^2\,dV,\;\; PE = \iiint_V \rho\Phi dV = \iiint_{VV} \rho gz\,dV$$

$$= \iiint_V (\rho_0(z) + \rho')gz\,dV \cong -\iiint_V \tfrac{1}{2}\,g\,\frac{\rho'^2}{\partial\rho_0/\partial z}dV = \iiint_V \tfrac{1}{2}\,g^2\,\frac{\rho'^2}{\rho_0 N^2}dV$$

where in the last line we neglect compressibility, and assume small deviations ρ’ from a basic vertically stratified fluid.

It is estimated (Oort, JGR ~1990) that the ratio of available PE (≡APE) to KE in the atmosphere is about 5 and in the oceans about 50. These numbers differ rather greatly from the simple estimate $(L/\lambda)^2$ which would be more like 10 for the atmosphere and 3000 for the ocean, if L is taken to be the size of the fluid region. The apparent overestimate is due to the presence of jets in both air and sea, which tend to dominate the volume integrated KE, despite their small volume. Also, the existence of a barotropic mode, with either velocity independent of depth (ocean) or fitting the eigenvalue structure of the stratified atmosphere, whose density vanishes at high altitude, implies KE without any APE.