Math Review/content for GFD-1 -- P.B. Rhines Rev. 1 2 iii 2004

If you open the classic applied math book (actually two books) *Methods of Theoretical Physics* by Philip Morse and Martin Feshbach at MIT, one of the first things you encounter in the discussion of vectors is .. vorticity. Vector fields are a common language of science, and the equations of fluid dynamics are prominent alongside Maxwell's equations, Schrödinger's equation, and classical dynamics.

There are many styles and levels of applied math text, some more attractive than others. Mathematicians sometimes look down on practical matters, writing 'special' books for engineers and more rigorous and abstract books for themselves. While there will always be some distance between mathematicians, physicists and geophysical scientists recent discoveries have done much to bring us back together. *Chaos* theory has been called the greatest discovery in classical physics of the last century. It came to prominence in part from the work of the meteorologist Edward Lorenz at MIT. The Lorenz equations are 3 coupled nonlinear o.d.e.s in 3 dependent variables, $x_1(t)$, $x_2(t)$, $x_3(t)$, which describe the oscillations and circulation of a circular loop of fluid heated from below (we produced one in the lab). In this 'hula hoop' the fluid can oscillate like a pendulum, with the cold fluid being denser than the warm, and it can circulate steadily round and round. The famous 'butterfly diagram' plots one of the variables against another (this is known as a phase-space plot), as time proceeds. The equations are easily solved on a laptop computer and are in the Matlab demo package. (for light reading see *The Essence of Chaos* by E.N.Lorenz; *Chaos* by J.Gleick)

Chaos theory teaches us that simple equations have complex behavior. In fact Richard Feynmann in his famous 3-volume lectures on basic physics uses the fluid dynamics equations as an example of this. The Lorenz equations have the property that, while they are 'determinisitic' (not random), and the solutions look periodic (like a pendulum, which they are), they can be completely non-periodic, slowly drifting from one oscillating state to another. Slight changes in initial conditions lead to great changes in solution, which is the basis of the theory of 'predictability' in weather forecasting. Chaos theory is sometimes oversold, as being the answer to all our problems and solving the problem of turbulent fluid motion: yet it is an exciting confluence of math, physics and atmosphere/ocean science in which everyone has learned from everyone else. If you think of the Lorenz equations as a fluid pendulum, interacting with an 'overturning circulation' you will understand it better than most people do!

What areas of applied maths are useful for GFD? Some of the principal research tools involve

solving fluid dynamics problems, using theory, numerical simulation and lab experiments;

time-series analysis and 'pattern analysis' of space-time fields (for example, doing EOF analysis of the North Atlantic Oscillation).

analyzing the small-scale physics of ocean and atmosphere, which involves physics, chemistry and even biology and geology

Some of the specific tools, heavy in math are:

vector calculus
trigonometry and geometry
Ordinary and partial differential equations
    waves: method of characteristics; (non-dispersive waves)
        stationary phase method, group velocity (dispersive waves)
        wave modes and rays
        resonance
Fourier analysis
Green's functions
Matrix algebra
asymptotic analysis: perturbation theory
    boundary layers
linear equations but with non-constant coefficients
    waves in a non-uniform medium
        ray theory (geometrical optics), refraction
        scattering
        Schrödinger's equation, potential wells and barriers
nonlinear dynamics techniques
    phase plane
    steepening, breaking, hydraulic flows
    wave interactions: resonant triad interactions
    momentum transport by waves, wave drag on topography
    interaction between waves and mean circulations
    geostrophic turbulence theory
statistical mechanics and thermodynamics

Experience has shown that one learns theory of this kind best when young, so try to do some coursework in these areas.

Vector calculus

1. Vectors and Cartesian tensors. Examples: velocity, momentum, stress.
2. Expressions for div, curl in vector and Cartesian tensor notation
3. Work out simplified expressions for

$$\nabla \bullet \nabla \times (\vec{A}),\ \nabla^2(\vec{A}), \nabla \times \nabla \times (\vec{A}), \vec{A} \bullet \nabla \vec{B} \ldots\ldots$$

Cartesian tensor notation helps greatly to work these outl

4. Line and area integrals. Divergence theorem (Gauss' theorem), Green's/Stokes' theorem, integration by parts. Relating vorticity and circulation, conservation and flux.

5. Relations with geometry: extrema, saddle-points

Ode's

1. forced oscillator, resonance
2. homogeneous and particular solutions.
3. Impulse response, Green's function (note biography of George Green in Physics Today, Dec. 2003)
4. Sturm-Liouville systems: eigenfunctions and eigenvalues.
5. Non-constant coefficients: harmonic form of Schrödinger eqn: trapping and tunneling.

Pde's

2. The linear, 2d order pde

$$a\psi_{xx} + b\psi_{xy} + c\psi_{yy} + d\psi_x + e\psi_y + f\psi = h \qquad (1)$$

describes the field ψ(x,y), if the right kinds of boundary conditions are applied. One of the independent variables (x,y) can be time, and there can be 1, 2, or 3 space variables (x,y,z) or (x,y,z,t)…. The coefficients a,b… are here constants, though in many applications they will be functions of the independent variables, space or time.

This equation is *linear* so that you can add two solutions together to get a third solution. This is the basis of Fourier analysis, solving for one sine-wave solution and adding up many such solutions to form a solution that is not very sinusoidal at all. The fluid dynamics equations are nonlinear in general, but many problems have a sensible linear version which can be understood, and then extended by inclusion of some nonlinear effects. Oceanic gravity waves and atmospheric Rossby waves propagate half-way round the Earth; the gravity waves are close to being linear, yet they interact with one another and break on the shore. Rossby waves are not such 'good' waves: they interact more quickly with one another and 'break' in the stratosphere. Yet, the simplest linear theory of Rossby waves is very useful and starts us on our way toward models of the atmospheric circulation.

The nature of the solutions of (1) depends on the sign of the discriminant,

$$b^2 - 4ac$$

of the most-differentiated terms. It sorts out as follows, with examples of each of the three cases:

For *negative values: elliptic* (example: Laplace equation, $\psi_{xx} + \psi_{yy} = 0$; electrostatic field, irrotational fluid flow); Poisson equation $\psi_{xx} + \psi_{yy} = b(x, y)$; Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi = 0$). Fundamental solutions:

$$\psi = \log(r) \ \ (2D\ Laplace\ eqn)$$
$$\psi = 1/r \ \ (3D\ Laplace\ eqn)$$
$$\psi = e^{-y}\sin(kx)\ (2D\ Laplace\ \ eqn)$$
$$\psi = \exp(ikx + ily)\ \ (2D\ Helmholtz\ eqn)$$

Laplace's equation is related to complex variable theory, because analytic functions of a complex variable have real and imaginary parts that obey Laplace's equation. Conformal mapping is possible, in which a solution for one shape of boundary is converted to a solution for another boundary shape by a mapping function. This gives a wealth of solutions which kept Victorian mathematicians busy for years.

Solutions of this equation cannot take on a maximum or minimum value away from a boundary. The solutions have the interesting property that they have the least energy (kinetic energy of the flow) of any conceivable flow with the same boundary conditions, so long as it obeys mass conservation. 'Lazy' solutions.

*positive values: hyperbolic (*example: classic wave equation, $\psi_{yy} - \psi_{xx} = 0$ or after a rotation of coordinates, $\psi_{xy} = 0$; vibrating string, with y=time; internal gravity waves with exp(-iσt) time-dependence: $\psi_{xx} - \psi_{zz} = 0$). The solution 'flows' along characteristic curves, with a definite 'domain of dependence' which is the region of x,y space that determine the solution at a given point.

Fundamental solutions (let y be the time, t, for clarity):

$$\psi = m(x+t) + n(x-t)$$
$$\psi = \exp(ikx - i\sigma t)$$
$$\psi = J_n(r)\exp(in\theta - it)\ \ (in\ cylindrical\ coords...Bessel\ \ functions)$$

Note that if we separate out the time-dependence, the classic hyperbolic wave equation becomes an elliptic Helmholtz equation. This says that wave comprised of a sine-wave in time multiplying a fixed spatial structure are 'modes' which have filled out the space and are no longer subject to the limited 'domain of dependence' ideas.

zero value: parabolic (example: heat equation, $\psi_{xx} - \psi_y = 0$). Fundamental solution:

$$\psi = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/4t)$$
$$\psi = e^{-y}\sin(x)$$

Boundary conditions: well-posed, over-specified, under-determined.

Boundary conditions are important: they can be the means of forcing the fluid into motion (or the fluid can be forced by a right-hand term, h in (1)). There is a very useful 'matrix' diagram that tells what sort of boundary condition goes with which equation. The classic theory involves boundary conditions that specify the value of the dependent variable ψ on the boundary, specify its normal derivative $\partial\psi/\partial n$, or a combination of both. The boundary can be 'open' lying on just one side of the fluid, or 'closed', that is, surrounding the fluid. This 'matrix' is attached (copied from Morse and Feshbach).

We find that hyperbolic equations, like the classic wave equation above, have a 'local' aspect, with solutions propagating along characteristic curves. Elliptic and parabolic equations are more 'global', with boundary conditions being felt everywhere. In the more complex equations of GFD we often ask this question: does the solution propagate along a few paths in the fluid or does it reach out an respond to boundary conditions everywhere? It's a good question to keep in mind when doing numerical model simulations (you can test them with this in mind, and very often you will have an open boundary to deal with: how does the missing information outside this virtual boundary affect your solution?

3. p.d.e. Solution techniques

homogeneous (unforced in interior):

- Separation of variables: reduction to a set of o.d.e.'s by simple choice of 'lateral coordinate' dependence
- plane-wave solutions (a special case of separation of variables)
  - complex notation: amplitude and phase of a wave
  - a few important results from complex variables
- Point-source solutions (Green's function)
- Reconstitution of more complex solutions using Fourier analysis/convolution
  - energy spectrum
  - thinking on the x-t plane (Hoevmueller plots)
- Method of characteristics

forced by a r.h.s. term:

- Green function, Fourier analysis

Curvature, slope, value. The second-derivative terms $\psi_{xx}+\psi_{yy}$ are the *curvature* of the surface $\psi(x,y)$, so long as its amplitude is small. This curvature may physically be a restoring force (as in a tense elastic membrane). Visualizing solutions to a Poisson equation $\psi_{xx}+\psi_{yy} = F(x,y)$ is aided by thinking of them as the deflection of a membrane due a weight and lift force distribution, $F(x,y)$.

Wave mathematics. Many GFD problems involve waves and their p.d.e.'s. Many of the methods above take on special character with waves: Green's functions, Fourier analysis, modes, perturbation theory. An example is WKB theory for waves propagating in a spatially varying medium; for example long gravity waves with a sloping bottom.
A key result, which can be established in a few lines is that

$$\psi_{xx} + \varepsilon^{-2} b(x)\psi = 0$$

where ε << 1, has solutions

$$\psi = const. \times b^{-1/4} \exp(i\int b^{1/2} dx)$$

which are waves with amplitude and wavelength both varying as the wave propagates through changes in b(x). The way to derive this is to assume a solution of the form

$$\psi = \exp(\varepsilon^{-1}\int n(x)dx)$$

and expand n(x) as an ordinary perturbation series, $n = n_0 + \varepsilon\, n_1 + \dots$
Substitute and equate terms multiplied by the same power of ε, and the result follows in 2 lines (it is given at the end of this document). It is an interesting idea: in ordinary perturbation theory we look for *small* corrections to a basic wave or flow; here we look for *slowly varying* corrections to a wave, which may be big if we follow them far enough.

WKB analysis can describe wave modes in a channel, on a sphere or as we see below, in the tropics. Total internal reflection makes 'modes' out of 'waves', as one hears with guitars and organ pipes. The energy and momentum and potential vorticity aspects of wave mathematics are exciting: for example waves propagating through mean circulation patterns do not conserve their wave energy: they tend to conserve 'wave action' which is energy/frequency (the frequency measured by an observer moving with the mean flow).

Equatorial waves. An important and useful example of a wave p.d.e. with spatially varying coefficient is the waves in a single-layer fluid along the Equator. Take y = 0 to be the Equator. The hydrostatic MOM equations combine to make an equation for a single variable..here the north-south velocity. At low latitude f ≅ βy, a linear approximation to 2Ω sin(latitude).

$$c_0^{-2}[v_{tt} + \beta^2 y^2 v - v_{xx} - v_{yy}]_t - \beta v_x = 0$$

(Gill p.435). There is a 'new' term here, βv_x, which comes when f(y) is differentiated in y. This north-south variation of the Coriolis frequency is what leads to Rossby waves.
With respect to x and t this is a hyperbolic p.d.e.; for simple-harmonic time and x-dependence, we separate variables, v = v(y) exp(ikx-iστ) and

$$v_{yy} + (\frac{\sigma^2}{c_0^2} - k^2 - \beta k/\sigma - \beta^2 y^2/c_0^2)v = 0 \qquad (2)$$

where c_0^2 = gH. In absence of rotation this gives the usual non-dispersive long gravity waves, $\sigma^2 = c_0^2(k^2 + l^2)$

in two dimensions with wave-vector (two scalar wavenumbers) (k, l). Here we can see that the term in brackets ( ) will be negative as |y| increases, leading to decaying exp(-|y|) solutions there. The bracketed term can be positive near the Equator. This is a potential well and Schrödinger equation, in which we fit oscillating (in y) solutions inside the well. The solutions are expressed in terms of classic Hermite polynomials,

$$v = 2^{-n/2} H_n((\beta / c_0)^{1/2} y) \exp(-\beta y^2 / 2c_0) \exp(ikx - i\sigma t)$$

$$\textit{with dispersion relation } (\sigma / c_0^{\,2}) - k^2 - \beta k / \sigma = (2n+1)\beta / c$$

There are two branches to the solution: one represents long gravity waves modified by rotation and trapped in the equatorial 'wave-guide',

$$\sigma^2 \cong (2n+1)\beta c_0 + k^2 c_0^{\,2}$$

It is valuable to look at these solutions as gradually varying sine waves and apply WKB ray theory, which explains most of the solution. Turning points occur where the curvature v_{yy} changes sign: the waves bounce along the equatorial waveguide.
The other branch represents Rossby waves at much lower frequency,

$$\sigma = -\beta k(k^2 + (2n+1)\beta / c_0)$$

There is also a 'rogue' mode, the Kelvin wave which is very similar to a coastal Kelvin wave: it has the dispersion relation eastward along the Equator of a simple non-rotating gravity wave, yet has a geostrophic balance north and south, giving a solution

$$\eta = \exp(-\tfrac{1}{2}\beta y^2 / c_0) \exp(ikx - i\sigma t); \;\; \sigma = +kc_0$$

The Kelvin and Rossby modes in the ocean are key components of el Nino/Southern Oscillation cycles. In the atmosphere they are key components of the quasi-biennial oscillation, Walker circulation, MJO….

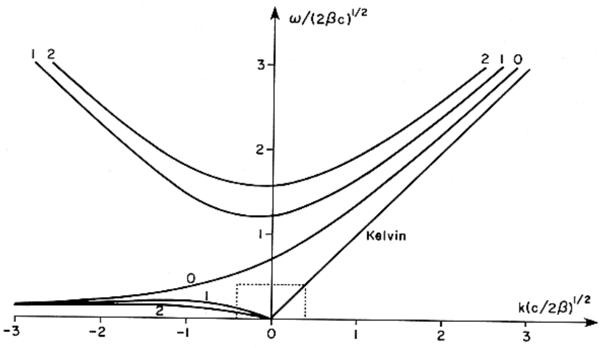


Fig. 11.1. Dispersion curves for equatorial waves. The vertical axis is the frequency in units of $(2\beta c)^{1/2}$ and the horizontal axis is the east–west wavenumber in units of $(2\beta/c)^{1/2}$. The curve labeled 0 corresponds to the mixed planetary–gravity wave. The upper curves labeled 1 and 2 are the first two gravity wave modes and the corresponding lower curves are the first two planetary wave modes. [Reproduced from "Numerical Models of Ocean Circulation," 1975, by permission of the National Academy of Science, Washington, D.C.]

Numerical solution

Any discussion of p.d.e.'s should talk about numerical solutions. Computer simulations are now a dominant part of our field (employing probably one-half the people in the field). Learning calculus, we define derivatives as a limiting process of taking finite differences of a function over an interval, and letting the interval become small. This is of course what a numerical model does, making the interval as small as we can afford. The finite difference models usually set up a regular, rectangular grid and develop the equations, with boundaries that wander through the grid and are treated approximately.

Density coordinate models use a regular grid in x and y, but follow a stack of density layers in the vertical, rather than remaining fixed in z. There are many advantages to this, because as we know, fluid tends to move along isopycnal or isentropic surfaces (-constant potential density) unless it is forced by heating, cooling or mixing.

Sigma-coordinate models account for the mountainous lower boundary using coordinate surfaces that follow the topography, and gradually become horizontal as one moves upward.

Fourier analysis suggests other ways to build a numerical model, letting the computer work in 'Fourier space', representing the pressure, velocity, density fields as finite sums of sine waves (as many as we can afford). The Fourier-based models can be quite elegant with very short codes, but they work best in very simply shaped domains.

Finite-element models are an exciting tool that can accommodate complex boundary shapes and topography. They are not yet used widely in O/A modeling despite their wide use in aerodyanamics, hydrology and other subjects. They have the virtue of variable grid resolution, fitting much resolution into narrow passages and around topographic features. This does, however, have a computational penalty.

Multi-grid methods involve having several grids at one time, some coarse and some fine. One often has to solve a Poisson equation for example, involving pressure; if this is done first on a coarse grid and then that solution is used to solve it again on a finer grid, it can be very efficient.

Here is a very simple exercise to get a feeling for the challenge of numerical solution of p.d.e.'s. Consider a finite difference model of a vibrating string. You can actually build it as a physical model, by clamping a series of lead fishing sinkers to a light nylon string. By concentrating the mass at discrete points we have essentially a finite-difference model of a smooth string. If ψ_n is the vertical displacement of the n^{th} lead sinker the vertical force on that sinker is proportional to the downward pull from the sinkers to the left and to the right of it

$$\begin{aligned}&T(\psi_{n+1}-\psi_n)-T(\psi_n-\psi_{n-1})\\&=T(\psi_{n+1}-2\psi_n+\psi_{n-1})\end{aligned}$$

where T is the tension in the string. This is recognizable as the finite-difference approximation to $\partial^2\psi/\partial x^2$. So the equation to be solved is

$$M\frac{d^2\psi_n}{dt^2} = T(\psi_{n+1} - 2\psi_n + \psi_{n-1})$$

Here M is the mass of each weight and T is the tension in the string. Try a sine-wave solution,

$$\begin{aligned}\psi_n &= \exp(i\alpha n - i\sigma t)\\ &= \exp(iKn\Delta x - i\sigma t)\end{aligned}$$

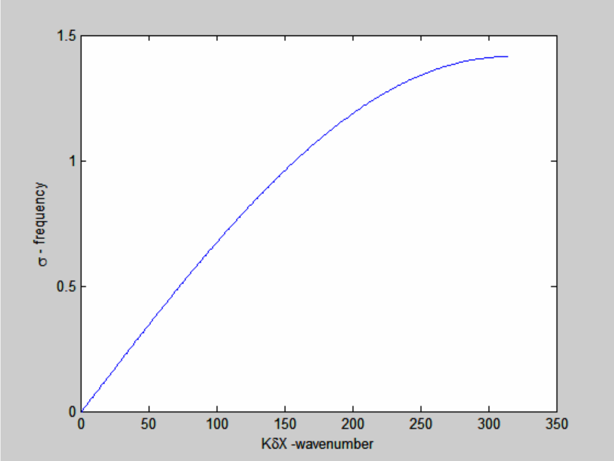
where α is a constant, which we rewrite as KΔx, K being a wavenumber and nΔx being a distance along the x-axis.

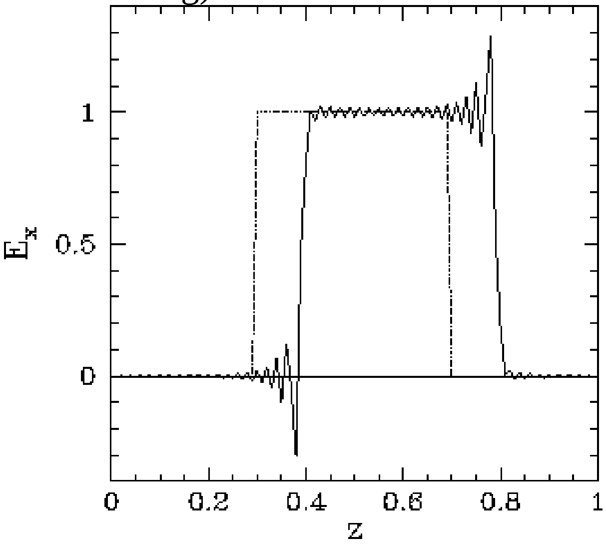
Substituting in the equation we find

$$\begin{aligned}\sigma^2 &= -(T/M)(e^{i\alpha} - 2 + e^{-i\alpha})\\ &= -(T/M)(2\cos(\alpha) - 2)\\ &= (2T/M)(1-\cos(K\Delta x))\end{aligned}$$

so $$\sigma = ((2T/m\Delta x)(1-\cos(K\Delta x)))^{1/2}$$

which is plotted below. We have written $m = M/\Delta x$ as the mass per unit length along the string. In the limit $K\Delta x \Rightarrow 0$, $\sigma = (T/m)^{1/2}K$. It shows the dispersion relation to start linearly, like that of the continuous string, but to fall off and become horizontal at the point where $K\Delta x = \pi$, which is where the wavelength $2\pi/K = 2\Delta x$. This is called the '2 delta x wave', and it is the shortest sine wave that can occur on this discrete string. The *group velocity* is the slope of the dispersion relation and it goes to zero there: the 2-delta-x wave does not propagate at all, and both phase speed and group speed of the waves is smaller than with a continuous string. So a square pulse, instead of propagating without change of shape as it does with the continuous, non-dispersive string, leaves a wake of short waves behind. The numerical model has introduced dispersion into the waves (of course it is a perfectly accurate description of a lead-fish-sinker string).





The righthand figure shows that a square pulse propagating to the right leaves short waves behind..which are a ‘wake of bad numbers’ if one is trying to model the continuous string. Further analysis of this model shows that the if we also finite-difference the time dependence, making a forward integration of the x and t numerical model, the solution may be stable or unstable: for the simplest form of the η_{tt} term, this depends on the ‘CFL parameter’, $c_0\Delta t/\Delta x$, which if too big, leads to an unstable solution.

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WKB solution for waves propagating through a varying medium (e.g. Bender and Orszag text). The properties of the medium appear in b(x) which we suppose varies over distances of order 1. The Helmholtz equation (after separating out exp(-iσt) time dependence) is

$$\psi_{xx} + \varepsilon^{-2} b(x)\psi = 0$$

where ε is a small parameter (being the wavelength divided by the length scale of variation of the medium, b(x). Look for a solution of the form

$$\psi = \exp(\varepsilon^{-1}\int m(x)dx)$$

The motivation behind this is that we are looking at waves (curvature of ψ, ψ_{xx}, returns ψ toward zero whether ψ is positive or negative). The *phase* of the waves is $\int m\, dx$, and m is the wavenumber. We know that the solution will not be a small correction to a uniform wave-train of constant amplitude and wavelength, where we might try

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon\psi_3 + ...$$

but it may be a *slowly varying* correction to one. Thus our perturbation series in effect looks a multiplicative corrections rather than additive corrections. First calculate ψ_{xx}:

$$\psi_{xx} = (\varepsilon^{-1} m_x + \varepsilon^{-2} m^2)\exp(\varepsilon^{-1}\int m\, dz)$$

The equation is

$$[(\varepsilon^{-1} m_x + \varepsilon^{-2} m^2) + \varepsilon^{-2} b]\exp(\varepsilon^{-1}\int m\, dz) = 0$$

Now expand m(x) as an ordinary additive perturbation series,

$$m = m_0 + \varepsilon m_1 + ...$$

substitute in the equation just above and group the term according to the power of ε.

$$\varepsilon^{-2}[m_0^2 + b] + \varepsilon^{-1}[dm_0/dx + 2m_0 m_1] + ... = 0$$

The idea of perturbation theory is that the answer should be valid for a wide range of values of ε, so we must equate each set of brackets [ ] to zero. We find

$$
\begin{aligned}
m_0 &= \pm i b^{1/2} \\
m_1 &= -\tfrac{1}{2}\frac{dm_0/dx}{m_0} = \frac{d}{dx}\log({m_0}^{-1/2}) \\
&= \frac{d}{dx}\log(i^{-1/2}b^{-1/4})
\end{aligned}
$$

So

$$
\begin{aligned}
\psi &= \exp(\varepsilon^{-1}\int m(x)dx) \\
&= \exp(\varepsilon^{-1}i\int b^{1/2}dx)\exp\varepsilon^{-1}\int\frac{d}{dx}\log(i^{-1/2}b^{-1/4})dx \\
&= (\pm i)^{-1/2}b^{-1/4}\exp(\varepsilon^{-1}i\int b^{1/2}dx)
\end{aligned}
$$

which is the WKB solution, showing how the amplitude and wavelength of a wave changes as it moves through a varying ‘index of refraction’. This is a very good way to find the vertical internal wave structure when the buoyancy frequency N varies with z, or Rossby waves on a sphere, or gravity waves with shoaling depth (they increase in amplitude and decrease in wavelength as they approach shore. Comparisons can be made with exact solutions (for which the ‘slowly varying’ assumption is not necessary). One little warning: if you derived your equation for constant properties (uniform depth, or uniform N) and want to study non-constant properties, don’t just let h or N vary in that equation. Go back and repeat the derivation with non-constant h or N, because there will be new terms there.

Below is a very useful chart from Morse and Feshbach, Methods of Theoretical Physics, vol I.

unique results for Dirichlet conditions on an open boundary if we are working in the positive direction along the characteristic but are unstable in the negative direction.

Physically this comes about because parabolic equations (the diffusion equation is an example) represent situations where the entropy is increasing as time increases. Therefore irregularities in the field ψ will tend to "iron out" as time increases; the sharper the irregularity, the more rapidly will it disappear. If we wish to work backward in time to find out what field, a minute (or an hour) before, eventually diffused into the specified distribution, we shall not be able to tell how many sharp irregularities there had been, at this earlier time, which had practically disappeared by the later time and thus did not show up to an appreciable extent in the specified distribution.

The results of the discussion of this section may be summed up in the following table:

Conditions	Boundary	Hyperbolic equation	Elliptic equation	Parabolic equation
Dirichlet or Neumann (Value or slope specified)	Open	Insufficient	Insufficient	**Unique, stable solution in positive direction, unstable in negative direction**
	Closed	Solution not unique	**Unique, stable solution** (see page 698 for Neumann conditions)	Solution overspecified
Cauchy (Value and slope specified)	Open	**Unique, stable solution**	Solution unstable	Solution overspecified
	Closed	Solution overspecified	Solution overspecified	Solution overspecified

The satisfactory combinations of equations and boundary conditions are those indicated by boldface type. We note again that Dirichlet-Neumann conditions may be *homogeneous* $[\alpha\psi(s) + \beta N(s) = 0]$ or *inhomogeneous* $[\alpha\psi(s) + \beta N(s) = F(s)]$. Homogeneous Dirichlet conditions mean that ψ will be *zero* on the boundary; inhomogeneous Dirichlet conditions mean that ψ will have specified nonzero values on the boundary, and so on.

6.3 *Eigenfunctions and Their Uses*

We have now reached a point where we must cease dealing with generalities and begin to get down to cases. We have spent the first