

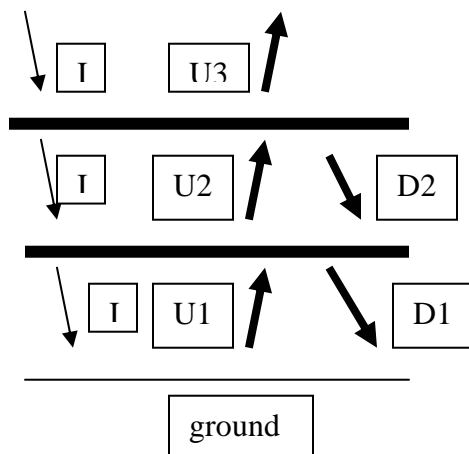
## GFD-1 Problem Set 1 OC-512/AS-509 SOLUTIONS

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1. The *greenhouse 'blanket'*. Carry out the greenhouse calculation in Gill, except use two plates of glass (thick lines on the diagram below), each of which absorbs (and re-radiates) a fraction  $\gamma$  of the long-wave infrared radiation reaching it, yet passes short-wave visible light with no reflection or absorption. Show that the increase in surface temperature at the ground can be greater than for a single glass plate absorbing a fraction  $\gamma$  of the incident radiation, even if  $\gamma=1$  in both cases. Recall that the outgoing longwave radiation on one side of a pane of glass is equal to the partially transmitted longwave plus  $\frac{1}{2}$  the absorbed longwave radiation ( $\frac{1}{2}$  because the absorbed radiation is re-radiated equally above and below). Set up the equations as in Gill p9, and find the upward longwave radiation in the lowest layer in terms of  $I$ , the incident short-wave solar radiation; hence if each pane of glass is a perfect absorber of IR ( $\gamma=1$ ), compare the temperature at the ground with the temperature with 1 pane of glass and with none.

(Hint, it seems easier to do this problem as sketched below, with just a single upward ( $U$ ) and a single downward ( $D$ ) longwave component in each layer (rather than defining extra components like  $B$  in Gill's fig. 1.3. Then for example the equation for the upward longwave radiation in the middle layer is  $U_2 = U_1(1-\gamma) + \frac{1}{2}\gamma U_1 + \frac{1}{2}\gamma D_2 =$  partially transmitted upward flux + re-radiated upward flux + re-radiated downward flux. We can see that the origin of the extra trapping by the 2d pane of glass is the downward component  $D_2$  which is absent with a single pane of glass).



Energy conservation for the three layers (for steady state) gives

1.  $I = U_3$

2.  $I + D_2 = U_2$

$$3. I + D_1 = U_1$$

The transmission laws give:

$$4. U_2 = U_1(1-\gamma) + \frac{1}{2} \gamma U_1 + \frac{1}{2} \gamma D_2 \quad (\text{middle layer...see above})$$

$$5. U_3 = U_2(1-\gamma) + \frac{1}{2} \gamma U_2 \quad (\text{above top glass})$$

$$6. D_1 = \frac{1}{2} \gamma U_1 + \frac{1}{2} \gamma D_2 \quad (\text{lowest layer})$$

Eliminate  $D_2$  from 4. and 2.

$$7. U_2(1 - \frac{1}{2} \gamma) = U_1(1 - \frac{1}{2} \gamma) - \frac{1}{2} \gamma I$$

Eliminate  $U_3$  and  $I$  from 5. and 1.

$$8. U_2(1 - \frac{1}{2} \gamma) = I$$

Eliminate  $U_2$  from 7. and 8.

$$U_1 = I (1 + \frac{1}{2} \gamma) / (1 - \frac{1}{2} \gamma)$$

which is the answer. For  $\gamma = 1$ , total absorption of longwave by either plate,

$$U_1 = 3I \quad \text{and} \quad D_1 = U_1 - I = 2I.$$

and so the ground temperature is elevated by a factor  $3^{1/4}$  compared with no atmosphere. For a single plate in Gill the result was  $U_1 = 2I$  and  $2^{1/4}$  warming factor. Notice that  $D_1$ , the downward longwave re-radiation from the 'clouds' is  $2I$ , twice the incident short-wave solar radiation. Compare this with the actual atmosphere, for which estimates (handout) are 168 watts/meter sq downward shortwave sunlight, 390 watts/meter sq upward longwave ( $U_1$ ) and 324 downward back radiation ( $D_1$ ).

2. *Hydrostatic pressure.* In a fluid at rest, the vertical momentum equation in Cartesian coordinates is  $\partial p / \partial z = -\rho g$ . In words, the pressure at a point is equal to the weight of the fluid overhead (per unit area), plus whatever the pressure is at the top of the fluid. Calculate the total force (pressure x area) on the bottom of a Coke bottle and show that is *not* equal to the total weight of the fluid in the bottle. Explain.

The Coke bottle has a narrow neck (unlike a cylindrical Coke can).

Assume it is open so the pressure at the surface is atmospheric,  $p_A$ . The pressure on the bottom is  $p_A + \rho g h$  where  $h$  is the height of the fluid column in the center. The force on the bottom is  $A(p_A + \rho g h)$ . The upward force by the atmosphere is  $A p_A$ , so the net force is  $\rho g A h$ . The volume of the fluid is less than  $A h$  (which would be the volume of a cylinder, not a bottle. For a cone-shaped Coke bottle the volume is  $1/3 h A$ , and the weight of the Coke would be  $1/3 \rho g h A \ll \rho g h A$  for example.

If the force due to hydrostatic pressure is greater than the weight, the extra force must come from the non-zero pressure on the sloping sides

of the bottle (which push down on the fluid). The pressure exerted on a fluid by a boundary is  $\iint p \hat{n} dA$  where  $\hat{n}$  is a unit vector normal to the boundary, pointing into the fluid. This has a vertical component at the sloping sides of the bottle.

This may seem silly but an interesting, related question is whether the pressure force on the surface of the Earth would equal the weight of the atmosphere if there were no motion. The answer is only 'yes' for a thin atmosphere. Imagine a planet with very thick atmosphere, like Jupiter; as you shrink the solid planet to a small billiard ball, it does not have to support the atmosphere...why?

3. *What is pressure?* A point mass with velocity  $+U$  in the x-direction and mass  $m$  hits a rigid wall and rebounds with velocity  $-U$ . Show that the time integral of the force exerted by the point mass on the block is  $2mU$  (this time-integrated force is known as the 'impulse'). If instead of rebounding elastically, the particle sticks to the wall, what is the time-integral of the force it exerts on the wall? In the first example the kinetic energy of the particle stays the same; what happens to the kinetic energy in the second example?

If many such particles hit a wall, say their density in space is  $n$  particles per cubic meter, so that they continually arrive and bounce off the wall, what is the time-averaged force they exert on the wall? [assume for simplicity they are equally distributed in space and hit the wall at equal time intervals].

This is a model of the pressure force exerted on a wall by 'point-mass' molecules of a dilute gas. The model shows that the pressure force is proportional to the mass density of the particles which we call density, and a quantity proportional to the kinetic energy (KE) of the molecules, which in turn we call 'temperature'. Pressure is also an expression of the *momentum flux* of the particles which is proportional to  $u^2$ .

{*Further discussion:* after the model is generalized to 3 space dimensions we find (Batchelor, Intro. to Fluid Dynamics, 1.7)

$$\text{force/area} = p = \frac{1}{3} \rho \langle u^2 \rangle$$

where the brackets  $\langle \rangle$  indicate the average value and  $u$  is the speed (=magnitude of the velocity). This becomes  $p = nkT = (k/m) \rho T = \rho RT$  where:  $\rho = nm$  is the average density in  $\text{kg m}^{-3}$ ,  $k = 1.381 \times 10^{-23}$  Newton  $\text{m}^0\text{C}$  is Boltzmann's constant and  $R$  is the gas constant defined as  $k/m$ . Generally the temperature of a gas is proportional to the translational kinetic energy of molecules plus rotation and vibration energies of the molecules. But, in the simplest case of point masses with no rotation or vibration, the average KE of a molecule is just  $\frac{1}{2} mu^2 = \frac{3}{2} kT \equiv \frac{3}{2} mRT$ . Thermodynamics texts often avoiding telling us just what temperature is! }

Integrate  $F = m du/dt$  to get  $m\Delta u = -2mu = \int F dt$ . If the particle sticks to the all it is just  $-mu$ .

Now suppose there are many particles, with spacing  $\Delta x$  in the  $x$  direction. For a line of particles (1 dimension) the number hitting the wall each second is  $U/\Delta x$ . Each hit will contribute  $2mU$  to the time-integrated force on the wall. The time integrated force will be a staircase curve, with its smooth average equal to  $(2mU)Ut/\Delta x$  or  $2mU^2t/\Delta x$ . So the time averaged force itself is just the time integral divided by  $t$ .

Now suppose the particles have the same spacing  $\Delta y = \Delta x$  and  $\Delta z = \Delta x$  in all directions. The number of particles per unit cubic meter is  $n = (\Delta x)^{-3}$ . The force per unit area, which we call pressure,  $p$ , is

$$p = (2mU^2/\Delta x) / \Delta y \Delta z = 2mU^2 / (\Delta x \Delta y \Delta z).$$

Now define the mass density of the fluid as  $\rho \equiv m(\Delta x)^{-3} \text{ kg m}^{-3}$ . So the average pressure  $p$  is

$$p = 2\rho U^2$$

which is 4 times the kinetic energy of the molecules (here they have only an  $x$ -component of velocity) and the density. For point-mass molecules this is a model equation of state, if we define temperature  $T$  to be the average kinetic energy of the molecules,

$$T = \frac{1}{2} m U^2$$

. Then

$$p = (4/m) \rho T$$

This is an introduction to the kinetic molecular theory of an ideal gas, but we have to (1) allow for a distribution of velocities (direction and magnitude) (Maxwell-Boltzmann distribution) in the three directions and (2) allow for molecules with vibrational and rotational motion, which add to the translational kinetic energy to make up the total microscopic mechanical energy of the gas. This is discussed in Batchelor's text, sec. 1.7. If the molecules have the same kinetic energy in all three directions, the result is  $p = 2/3 \times$  kinetic energy of translation of molecules in a unit volume, and

$$p = \rho RT$$

where  $R = k/m$  is the gas constant for a particular gas. The final question of the prediction of the relation between the internal energy  $E$  and the temperature is somewhat less certain: there is not perfect agreement of experiments with the prediction from classical Newtonian laws.

The important thing is to understand pressure as a large-scale response to molecular impacts, and temperature as at least related to, if not

exactly equal to, the mechanical energy of those molecules. In a liquid there really is no simple kinetic molecular theory, with molecules separated by a few molecular diameters at most. But the same effect is working.

This is a discussion of the force felt by a wall, perpendicular to it, which equals the momentum flux of molecules in that same direction. Those molecules also have a momentum flux *parallel* with the solid boundary, and exerts a shear-stress (viscous force) on the boundary.

Inside the fluid, far from boundaries, the same kind of molecular motion exists and the pressure and viscous stress transfer momentum from one fluid lump to the next. The same argument given above applies with slight modification. Instead of a wall reflecting particles we have particles passing an imaginary plane in both directions; this gives the same connection between x-momentum flux and density and temperature. It is the basis for thinking about sound waves, incompressible fluid pressures, viscous forces and diffusion of other properties, from the molecular point of view.

4. Bobbing buoys. A spar buoy is a thin cylinder (like a log) which is floating vertically. If its height is  $H$ , cross-sectional area is  $A$  and mass density is  $\rho_B$ ,

- calculate the equilibrium position when it floats in water of density  $\rho_W$ . It may seem strange, but for some values of  $\rho_B/\rho_W$  the buoy will only be stable floating vertically, and for other values it floats horizontally (any idea how this works?).

- Now suppose the buoy is pushed down so that it oscillates (bobs) up and down; assuming that the pressure is hydrostatic, find the equation of motion in terms of the vertical displacement  $\eta(t)$ , and find the frequency of oscillation.

- if the buoy is shaped like a cone instead of a cylinder, how does the vertical restoring force depend on  $\eta$ ? In the language of classical mechanics, a 'hard spring' is one whose Hooke's law coefficient increases with increasing  $|\eta|$  while a 'soft spring' is one where it decreases.

Floating will occur when the immersed depth  $h$  obeys

$$h\rho_W = H\rho_B$$

The upward force, by Archimedes principle, is equal to that experienced by the water that 'was there before the buoy arrived', and this is  $g\rho_W$  times the volume of the immersed region. If the buoy is pushed down a distance  $\eta$ , that upward force  $F$  exceeds its weight

$$F = \rho_w g A \eta$$

and this causes a rebounding acceleration,  $d^2\eta/dt^2$ . The oscillation equation is thus

$$A h \rho_B d^2\eta/dt^2 + \rho_w g A \eta = 0$$

and we solve with solutions  $\eta = \text{const} \times \cos \sigma t$ , giving for the frequency

$$\sigma^2 = (g/h) \rho_w / \rho_B .$$

This all assumes that hydrostatic pressure is close enough to the true pressure, which will be true if  $\sigma$  is small enough...although there are other aspects of the drag force of the water on the buoy which would need to be looked at.

If the cross section of the buoy is not cylindrical but conical (say, with the point downward), depressing the buoy by a vertical distance  $\eta$  leads to a restoring force which is more complicated. The volume of a cone is  $1/3 \alpha^2 \pi h^3$  if  $\alpha$  is the angle between the cone axis and its sloping side. The immersed volume changes by an amount

$$1/3 \alpha^2 \pi (h^3 - h_0^3)$$

where  $h_0$  is the equilibrium depth of the cone. The restoring force increases faster than  $h$  when pushing down, yet slower than  $h$  on the up-stroke. The equation of motion is

$$m d^2h/dt^2 + 1/3 g \rho_B \alpha^2 \pi (h^3 - h_0^3) = 0$$

which is a *nonlinear* oscillator equation.  $m$  is its mass. For very small amplitude we can linearize the equation. Let  $h-h_0 = \eta$ .

$$m d^2\eta/dt^2 + g \rho_B \alpha^2 \pi h_0^2 \eta = 0$$

which is a simple oscillator equation, neglecting terms of order  $\eta$ .