

Geophysical Fluid Dynamics I P.B. Rhines

Problem Set 2 -----SOLUTIONS *

out: 22 Jan 2004

back: 29 Jan 2004 *note change in formula for velocity field in problem 1, and sign of ψ in problem*

4

1. Geostrophic flow. A vortex with circular streamlines can have velocity field of many different kinds; the tornado vortex in the lab has approximately $u_\theta = A/r$ azimuthal velocity ('swirl' velocity) which has uniform angular momentum. But this goes to ∞ at $r=0$, so a good model of such a vortex is

$$u_\theta = U_0 \frac{r}{a} \quad r < a$$

$$= U_0 \frac{a}{r} \quad r \geq a$$

Sketch the velocity as a function of r .

Solve for and plot the pressure field that balances this velocity

(a) without Coriolis effects ($\Omega=0$)

(b) with Coriolis effects ($\Omega \neq 0$)

Discuss the momentum balance in the radial direction, showing how it depends on the Rossby number, which here can be defined as $Ro = U_0/2\Omega a$. You will need to write the momentum equation in polar coordinates (or see Batchelor's text or others), dropping the time-varying terms ($\partial/\partial t \dots$) and frictional terms. Consider both cyclonic and anticyclonic vortices (U_0 positive and negative, respectively). Describe how the correspondence between 'cyclonic' and low pressure; 'anticyclonic' and high pressure depends on Ro .

Let u be radial velocity and v azimuthal velocity; radial momentum (r direction, v velocity) balance is

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{u}{r} \frac{\partial v}{\partial \theta} - \frac{u^2}{r} - 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad (\text{e.g., Batchelor p 603})$$

for a steady vortex, $u=0$, and

$$-\frac{u^2}{r} - 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

with the above velocity field, the r -integration gives

$$p = p(r = a) + \frac{1}{2} \rho U_0^2 \left(1 + \frac{2}{Ro}\right) \left(\frac{r^2}{a^2} - 1\right) \quad r \leq a$$

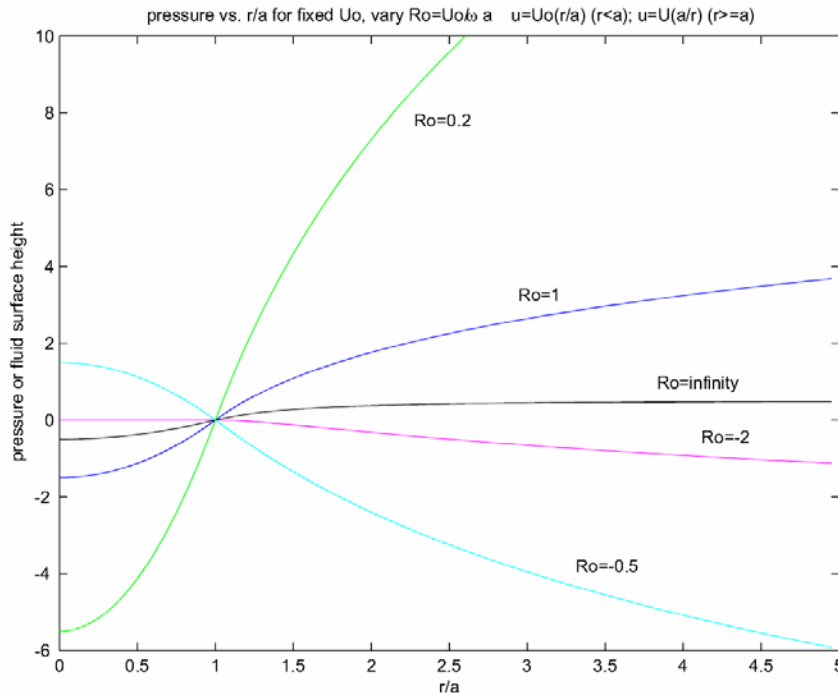
and

$$= p(r = a) + \int_a^r \rho U_0^2 \frac{a^2}{r^3} dr + \int_a^r \rho 2\Omega U_0 \frac{a}{r} dr \quad r > a$$

$$= p(r = a) + \frac{1}{2} \rho U_0^2 \left(1 - \frac{a^2}{r^2}\right) + \rho 2\Omega U_0 a \ln(r/a)$$

$$= p(r = a) + \frac{1}{2} \rho U_0^2 \left[\left(1 - \frac{a^2}{r^2}\right) + \frac{4}{Ro} \ln(r/a)\right] \quad r > a$$

where $Ro = U_0/\Omega a$. A bit messy, but just varies like r^2 inside and $-r^{-2} + \ln(r)$ outside. It is a nice shape..like the free surface observed in the lab. You find that the cyclones ($U_0 > 0$) have low pressure centers accentuated by rotation, whereas anticyclones ($U_0 < 0$) have high pressure if Ω is big enough (Ro is small enough).



The center pressure deepens with increasing a or increasing Ω or increasing U_0 or decreasing Ro . With $U_0 < 0$ an anticyclonic has high pressure at the center if $Ro < \frac{1}{2}$. The pressure as $r/a \Rightarrow \infty$ goes to a constant c_2 with no Coriolis effects, yet increases gradually like $\ln(r/a)$ if Ω is not zero. This suggests that the $1/r$ style vortex is not really totally 'local' or 'isolated' (it has equal angular momentum in each ring). If we used an $\exp(-r)$ velocity at large r/a this would be more isolated with zero integrated vorticity. Interestingly, the area integral of the vorticity is $2\pi a U_0$ for our vortex. In general the vertical vorticity of a circular flow $u(r)$ is

$$\frac{(ru)_r}{r} \equiv u_r + u/r$$

and its area integral is just $2\pi[ru]_0^\infty$.

2. Write a one-page essay on one of the lab experiments you have seen, taking it as far as you can: stating the experimental situation, the question in mind, what you observed, and what the implications to the atmosphere or ocean may be.
3. Large scale flows, if slowly varying in time, tend to be in geostrophic balance,

$$2\vec{\Omega} \times \vec{u} = -\nabla p / \rho + O(Ro) + O(1/\Omega T)$$

in the horizontal and hydrostatic

$$\frac{\partial p}{\partial z} = -\rho g$$

in the vertical. Consider the Antarctic Circumpolar Current (ACC) which travels eastward round the Southern Ocean. It has an average volume transport of roughly $140 \times 10^6 \text{ m}^3 \text{ sec}^{-1}$. (140 Sverdrups), and is largely wind-driven, by the westerly winds overhead.

Suppose the ACC is 1000 km wide, and the eastward velocity is uniformly distributed over this width and over a depth range of 2000m.

- Find the shape of the sea surface for the case of hydrostatic, geostrophic balance.
- Imagine the wind starting to blow round this belt (centered at 60° S latitude) with the ocean at rest. How would you expect the sea surface to change from its initially flat state? Ignore the presence of continents (this is an 'aqua-planet').

The zonal velocity is $u = \text{volume transport/area} =$

$$140 \times 10^6 / (10^6 \text{ m} \times 2 \times 10^3 \text{ m}) = 0.07 \text{ m sec}^{-1} = 7 \text{ cm sec}^{-1}$$

With $2\Omega \sin \phi u = -g \eta_y$ the profile of the sea surface will be

$$\eta = -(f/g) (0.07) (y-y_0) \quad f=2\Omega \sin \phi_0$$

if we assume constant Coriolis frequency, or

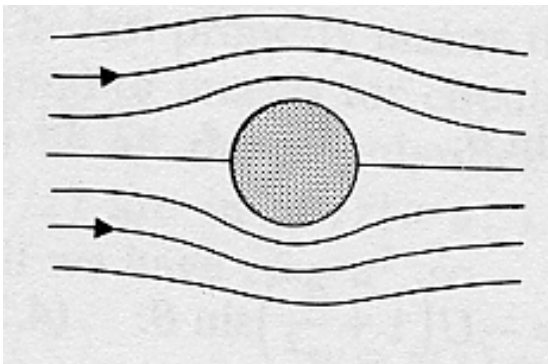
$$\eta = (2\Omega/g)(0.07) a \cos \phi$$

if we take account of the variation of f . The sea surface tilts up to the north (f is negative in Southern Ocean), with height difference from

one side to the other of $f(y-y_0) U/g \sim 0.7 \text{ m}$ (actually it is more than this because the current is more concentrated in the upper ocean). On an aqua planet this current would have a jump in η at the edges (we are not in a channel with walls). There, the geostrophic relation suggest a counterflow, westward u , and quite intense. In fact geostrophic balance says that the integral of the surface current is proportional to the difference in η at the end-points (for constant f).

4. Consider a flow around a cylinder: this is a classic fluid dynamics problem. If the fluid has zero vorticity ('irrotational flow'), the solution for the streamfunction, ψ , is

$$\psi = -U \sin \theta (r - a^2/r)$$



which is the sum of a uniform flow in the x-direction plus a dipole source-sink distribution. See, e.g. Acheson, Elementary Fluid Dynamics, sec 4.5. Use Bernoulli's equation to find the pressure field, assuming the flow far upstream has uniform pressure. This part can be found in any basic fluid dynamics text.

Now add rotation: as shown in class, the velocity field can still be the same, if Coriolis forces are balanced by an added pressure field. Solve for this new pressure field, by assuming that far upstream the flow is geostrophic with $p = \rho f \psi$ for $x \Rightarrow -\infty$ (f is the Coriolis frequency, $2\Omega \sin \phi$). Sketch the result, showing the transition from a strongly rotating to a weakly rotating flow, as a function of the Rossby number which here can be defined sensibly as U/fa .

The procedure here is to find the new component of pressure (due to rotation) at a point r, θ by finding the value of ψ at that point, and back-tracking to $x = -\infty$ where that value of ψ corresponds to a known value of p .

Bernoulli's equation is not affected by gravity in this 2D flow, so it is just the conservation of $p/\rho + \frac{1}{2}(u^2 + v^2)$ following a streamline. Using the above result for ψ , we get

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2 \left[2\left(\frac{a}{r}\right)^2 \cos 2\theta - \left(\frac{a}{r}\right)^4 \right]$$

where we use the trig relation $\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta$
; $\sin 2\theta = \frac{1}{2}(1 - \cos 2\theta)$. There are other equivalent expressions like

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{1}{2}U^2 \left[\left(1 + \frac{a^2}{r^2}\right)^2 \sin^2 \theta + \left(1 - \frac{a^2}{r^2}\right)^2 \cos^2 \theta - 1 \right]$$

On the cylinder boundary $r=a$ this is

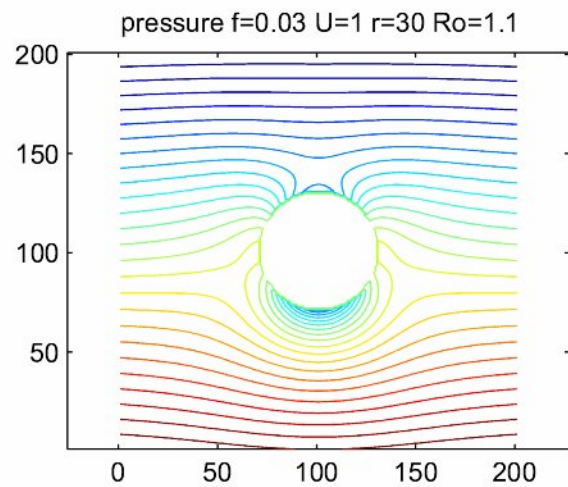
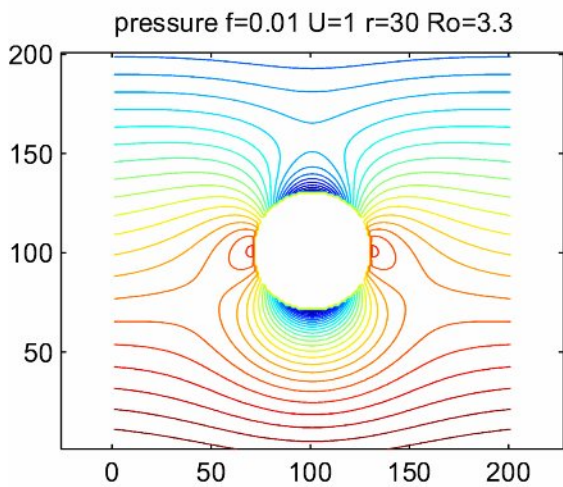
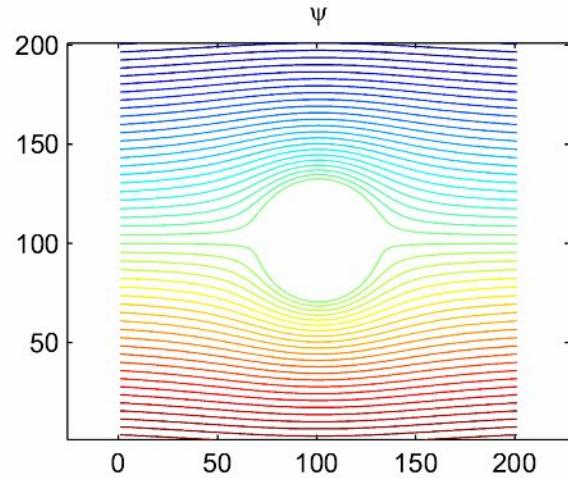
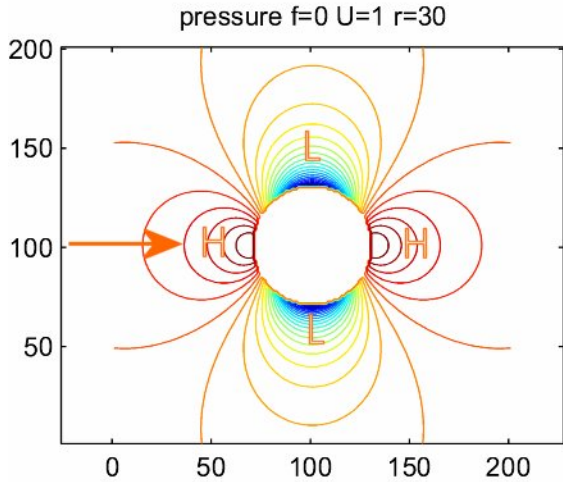
$$\frac{p}{\rho} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2(2\cos 2\theta - 1)$$

which shows that the pressure is high at the stagnation points $\theta = \pm\pi$ where $p - p_\infty = \frac{1}{2} \rho U^2$ and low at the top and bottom $\theta = \pm \frac{1}{2} \pi$ where $p - p_\infty = -3(\frac{1}{2} \rho U^2)$

The effect of rotation is to add a pressure component with isobars parallel to the streamlines. The flow is still a solution of the equations of motion without change (this is a rare property of 2D flows). This extra pressure is thus in geostrophic balance with the velocity. We want to find a function $p_r(\psi)$ that obeys the upstream boundary condition, which now does not have a uniform pressure but instead $p = \rho f \psi = -\rho f U y_\infty$ where y_∞ is the value of y as $x \Rightarrow -\infty$. This suggests that the function we are looking for is simply $\rho f \psi$ which equals $-\rho f U (r - a^2/r) \sin \theta$. The complete solution now is

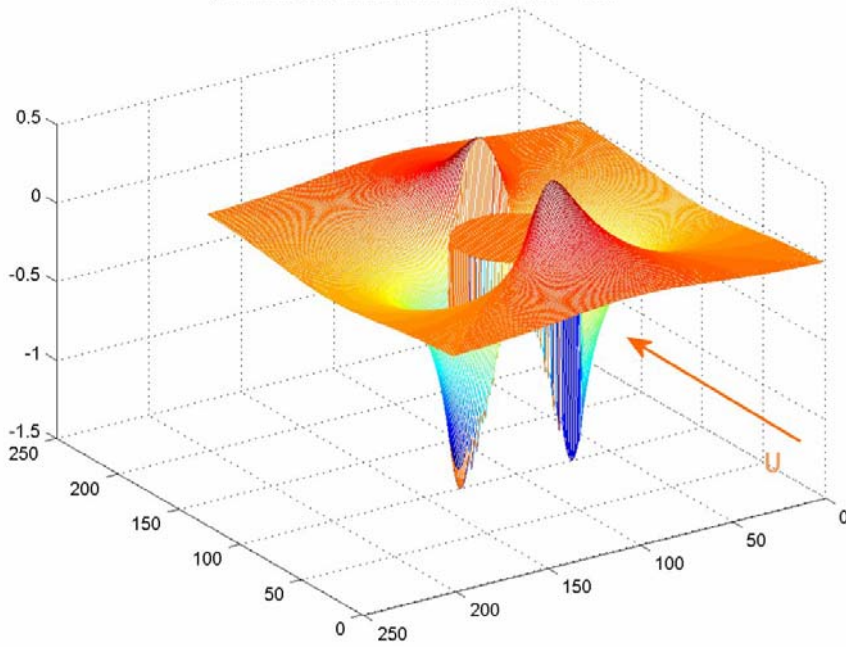
$$\frac{p}{\rho} = +\frac{1}{2}U^2 \left[2\left(\frac{a}{r}\right)^2 \cos 2\theta - \left(\frac{a}{r}\right)^4 \right] - fU \left(r - \frac{a^2}{r}\right) \sin \theta$$

which does go to $-\rho f U y$ far upstream. The effect is to add a pressure that is high to the south and low to the north, to the quadrupole shape of the pressure without rotation. With large f , the isobars almost lie along the streamlines but the small difference provides the acceleration needed along the streamline.

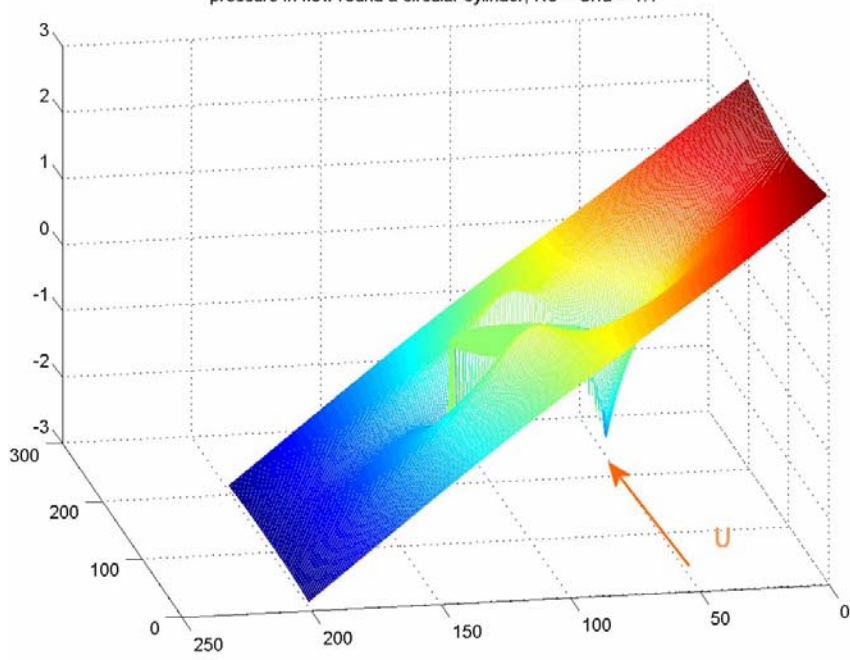


see perspective views of pressure field below. No matter how strong the rotation, the pressure variations along streamlines are essential to the acceleration of the fluid as it rounds the cylinder.

pressure in flow round a circular cylinder, $Ro = U/fa = \infty$



pressure in flow round a circular cylinder, $Ro = U/fa = 1.1$



pressure in flow round a circular cylinder, $Ro = U/ra = 0.33$

