Geophysical Fluid Dynamics I P.B. Rhines SOLUTIONS---

Problem Set 3 (revison 1)
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1. Use the energy equation (Gill 8.2) for the one-layer model of a wind-driven flow in a zonal channel to estimate the time for the wind-driven channel flow to develop, without using the time-dependent exact solution. To do this, calculate the KE and APE in the final steady solution, and write down the rate of energy input by the wind-stress (the force F(y)).

The ratio [total energy]/[rate of energy input] has the dimensions of time, and is an estimate of the time required to spin up the circulation. Note the general dependence on the energy ratio APE/KE on the scale of the circulation, L ($=l_0^{-1}$ where l_0 is the y-wavenumber) and the Rossby deformation radius.

The kinetic energy in the final solution is

$$\tfrac{1}{2}\rho H u^2 = \tfrac{1}{2}\rho H \frac{F_0^2}{R^2}\sin^2 l_0 y$$

per unit area. The available potential energy is

$$\tfrac{1}{2}\rho g \eta^2 = \tfrac{1}{2}\rho \frac{f^2 F_0^2}{g R^2 l_0^2}\cos^2 l_0 y$$

Their sum is $$\tfrac{1}{2}\rho H \frac{F_0^2}{R^2}[\sin^2 l_0 y + \frac{1}{\lambda^2 l_0^2}\cos^2 l_0 y] \qquad \lambda \equiv \sqrt{gH}/f$$

and the average across the channel is just

$$\tfrac{1}{2}\rho H \frac{F_0^2}{R^2}\tfrac{1}{2}[1 + \frac{1}{\lambda^2 l_0^2}]$$

Note that the ratio of APE to KE is the square of the Rossby radius divided by the square of the length scale of the flow. The rate at which the external force does work on the fluid is force x velocity x depth which ranges from 0 at the start to $F_0^2 \sin^2 l_0 y \;\; H^2/R$ at the end (the force F in the equations is force per unit depth). The ratio (total energy/rate of energy input) is thus (using the average value of ½ of the $\sin^2$)

$$\frac{1}{R}\tfrac{1}{2}\left[1 + \frac{1}{\lambda^2 l_0^2}\right]$$

For the interesting case of a channel much wider than the Rossby deformation radius (recall that the width of the channel is π/l_0) the estimate of the spinup time is

$$[R\lambda^2 l_0^2]^{-1}$$

which is just $\lambda^2 l_0^2$ times the normal frictional spin-up time, R^{-1}.

2. *Forced motion of a single-layer fluid: a local approximation.* In the rotating channel calculation in lectures we threw out some terms in the governing equation for free-surface displacement, η, in order to focus on low-frequency (long time-scale) dynamics. Let us revisit the problem, and think about the response of that fluid shortly after the wind-stress forcing is turned on. At early times the tilt in the fluid surface, $\nabla\eta$, is small. Therefore let us neglect the horizontal pressure gradient and simply solve

$$u_t - fv = F - Ru$$
$$v_t + fu = -Rv$$

In the original problem F is a function of y, but here take it to be a constant (we are looking at the local dynamics in a particular region, before the adjacent regions communicate by signaling through the η-field). Assume that F is switched on at time t=0:

$$u(t=0) = 0 = v(t=0)$$
$$F = 0 \quad (t < 0)$$
$$= F_o \quad (t \geq 0).$$

Solve for u(t), v(t), and particularly look at the oscillating and steady parts of the solution (again, homogeneous and particular solutions), and compare with the theory for the full channel problem (especially the direction the fluid moves in the mean from looking at v/u).

Homogeneous solutions: *NOTE: in an email we suggested that for the homogeneous solution, setting R=0 for this part would give enough of the full solution…the general case with R≠0 is given here. For the steady part we want to keep R non-zero to see its effect on the mean direction of the flow.*

$$u^h{}_t - fv^h = -Ru^h$$
$$v^h{}_t + fu^h = -Rv^h$$

$$u^h = U_0 \exp(-i\sigma t), v^h = V_0 \exp(-i\sigma t)$$
$$-i\sigma U_0 - fV_0 = -RU_0$$
$$-i\sigma V_0 + fU_0 = -RV_0$$

with the solution

$$\sigma^2 + 2iR\sigma - (R^2 + f^2) = 0$$
$$\sigma = -iR \pm \sqrt{-R^2 + (R^2 + f^2)}$$
$$= -iR \pm f$$

+

so the homogeneous solution is

$$u^h = \mathrm{Re}\,al(U_0 \exp(-ift)\exp(-Rt) = [A\sin(ft) + B\cos(ft)]\exp(-Rt)$$
$$v^h = \mathrm{Re}\,al(V_0 \ \exp(-ift)\exp(-Rt) = [C\sin(ft) + D\cos(ft)]\exp(-Rt)$$

which is exponential frictional decay on top of an inertial oscillation. The amplitudes are given by

$$\frac{U_0}{V_0} = i$$

so that we have basic inertial oscillations, with u and v being 90^0 out of phase in time, hence fluid particles moving on anticyclonic circular paths (the whole fluid moves as a rigid sheet with all the fluid particles doing the same thing). {By the way, a slightly easier way to solve the above problem is at the outset to define new variables $\tilde{u}(t) = u(t)\exp(-Rt);\ \tilde{v}(t) = v(t)\exp(-Rt)$ after which the equations for $\tilde{u}, \tilde{v}$ do not have the –Ru, -Rv terms and are very simple to solve.}

The particular solution, for steady forcing F, is found by dropping the time-differentiated terms to give

$$-fv^p = F - Ru^p$$
$$fu^p = -Rv^p$$

and the solution to this is the vector horizontal velocity,

$$(u^p, v^p) = \frac{(RF, -fF)}{f^2 + R^2}$$

So the steady component of forced flow moves in a direction θ defined by

$$\tan\theta = v/u = -R/f$$

that is, at 90^0 to the forcing with zero friction (at a speed independent of R if it is small), yet in the same direction as F for strong friction, R/f >>1, with speed simply F/R.

Now to satisfy the initial conditions, let the sum

$$u^h + u^p = 0, v^h + v^p = 0 \ \ (t = 0)$$

The result is

$$B = -u^p, D = -v^p$$

We need **another** relation to solve for A and C (the set of equations is 2d order in time, hence requires two initial conditions). With u=0, v=0 at the beginning, t=0, the momentum equations show that

$$u_t = F;\ \ v_t = 0$$

at that time. This says that for very early times, the fluid simply accelerates in the direction of the force. Therefore, working these out in terms of the u^h,v^h solution

$$fA - RB = F;\quad fC = RD$$
$$A = F/f - (R/f)u^p;\ \ C = -(R/f)v^p$$

and the full solution is

$$u = \frac{RF}{f^2 + R^2}\left[1 + [\frac{f}{R}\sin\ ft - \cos\ ft]\exp(-Rt)\right]$$

$$v = \frac{-fF}{f^2 + R^2}\left[1 - [\frac{R}{f}\sin\ ft + \cos\ ft]\exp(-Rt)\right]$$

At small time, ft <<1 you can verify that u ~ Ft and v ~ 0 (+ order t^2). The oscillations die away at large time, leaving the steady solution u^p,v^p. Note the solution for zero friction, R=0:

$$u = \frac{F}{f}\sin ft, \quad v = -\frac{F}{f}(1-\cos ft)$$

3. **'***Stiffening' of the fluid by the Coriolis force..and inertial waves.* The Earth's rotation limits the strength of circulations of ocean and atmosphere. It also 'stiffens' the fluid along the direction of the rotation vector (here vertical). In this problem we will relax the shallow-water, hydrostatic 2-dimensional restriction and look at forced, linear motion of a constant-density rotating fluid without a free surface.

$$\begin{aligned} u_t - fv &= -p_x/\rho \\ v_t + fu &= -p_y/\rho \\ w_t &= -p_z/\rho - g \\ u_x + v_y + w_z &= 0 \end{aligned}$$

becomes, with no y-variation:

$$\begin{aligned} u_t - fv &= -p_x/\rho \\ v_t + fu &= 0 \\ w_t &= -p_z/\rho - g \\ u_x + w_z &= 0 \end{aligned}$$

We can combine these 4 equations in 4 dependent variables into one equation for any one of u, v, w, or p. The fluid lies between z = 0 and z = -H. The result is

$$(w_{xx} + w_{zz})_{tt} + f^2 w_{zz} = 0$$

which is an equation for inertial waves.

- solve for a general homogeneous solution in the form of a wave, with frequency ω, and wavevector (k,m) […that is x-wavenumber is k, z-wavenumber is m].

  w = Real(B exp (ikx + imz – iωt)

The dispersion relation that results from substitution back into the wave equation is

$$\omega^2 = \frac{f^2 m^2}{k^2 + m^2}$$

The amplitude B is arbitrary at this point.

- plot or sketch the resulting dispersion relation, ω = function of k and m.
- now force the motion by imposing a vertical motion at the top of the fluid. Let w(z=0) = A cos(k_0x - ω_0t), and w(z=-H) = 0. Solve for w(x,z,t) by writing down the general homogenous solution and then applying both the boundary conditions. It is clear that we will choose k = k_0 and ω = ω_0 so that substituting back in the equation will give us the vertical wavenumber, m. Then the boundary conditions also give the amplitude, B. Note that you can readily satisfy the lower boundary condition by choosing w to vary in z like sin(m(z+H)).

This is a problem with two boundaries in which we can expect that if waves are involved there will have to be upward and downward propagating waves which will add up to make a standing wave in z (as one sees in the horizontal with water waves in a closed basin). You can form this solution by adding two plane waves of the kind written above, or a good procedure is to try separation of variables in the form w = r(z) x horizontally propagating part which fits the boundary condition

Try w =r(z) cos(k_0x - ω_0t) so as to fit the boundary condition. The equation for r(z) is then

$$-\omega_0^2(r_{zz}-k_0^2 r)+f^2 r_{zz}=0$$

$$r_{zz}+[\frac{k_0^2}{f^2/\omega^2-1}]r=0$$

This can be solved in terms of upward and downward exp(±imz) traveling waves, but by choosing sin(m(z+H) form of solution, the lower boundary condition is satisfied. Then the solution is

$$w=B\sin(m(z+H)\cos(k_0 x-\omega_0 t)$$

which satisfies the upper boundary condition if

$$B=\frac{A}{\sin mH}$$

where m is the solution of $m^2=k_0^2/(f^2/\omega_0^2-1)$. This solution is a 'standing wave' in the vertical z-direction and a traveling wave in the horizontal x-direction. Notice the denominator which can vanish for special values of the forcing frequency and x-wavenumber: this is where the fluid is forced at resonance making very large amplitude waves.

- consider the limit ω_0 << f. We will find that: the kinetic energy in the fluid becomes large, the v-velocity becomes much larger than the u-velocity or the w-velocity, and the horizontal velocity v becomes nearly independent of z (the fluid moves almost in columns), the pressure force exerted by the upper boundary condition becomes large, the v-velocity comes close to geostrophic balance and basically, the fluid becomes 'stiff'. Calculate at least 3 of these results and make a sketch of the velocity field in this limit.

At low frequency, ω << f, the dispersion relation gives m/k = ω/f <<1 which means that **the motion is organized in nearly vertical columns…Taylor columns.** The wave-crests are nearly vertical and the velocity nearly horizontal, which fits with the idea of columnar motion.

The equations above show that in this same limit the flow is nearly in geostrophic balance in the x-momentum equation (-fv ~ -p_x/ρ). In this limit

$$w=A\frac{\sin(m(z+H)}{\sin(mH)}\cos(k_0 x-\omega_0 t)$$

$$\approx A(\frac{z}{H}+1)\cos(k_0 x-\omega_0 t)$$

That is, w is a linear function of z. The x-velocity is found from mass conservation,

$$u_x=-w_z => u\approx -A\frac{1}{k_o H}\sin(k_0 x-\omega_0 t)$$

which is independent of z…for ω_0 << f. So scale analysis of estimates of u and other variables, are

$$u \sim \cot(mH)(m/k)w \sim (1/kH)w;$$

$$v \sim (f/i\omega)u = (\frac{f}{\omega\, kH})w;$$

$$kinetic\ \ energy \sim (\frac{f}{\omega\, kH})^2 w^2$$

$$p \sim (\rho f/k)v \sim \frac{\rho f^2}{\omega k^2 H} w$$

This shows that the pressure and v get very large if f is large, ω is small, H is small and/or the horizontal scale 1/k is large.
We would have the wrong answer if we had estimated w_z as $w_z \sim mw$….because w is such a small piece of a sine-wave, this estimate is not accurate. Due to the smallness of mH, the estimate of w_z is w/H which is much larger than mw.
Here the ~ sign (sometimes called 'twiddles') indicates the approximate magnitude rather than a statement of equality. Note that with mH << 1 the vertical velocity varies linearly in z, and the scale analysis needs to recognize this through the factor cot(mH) (~1/mH) above.

The kinetic energy is dominated by the v-velocity normal to the x,z plane of the 'wave'. It is $(f/\omega)^2$ larger than the w^2 component corresponding to the vertical motion at the upper boundary. v/u ~ f/ω, and the pressure becomes very large as ω/f becomes small (the upper boundary has to push and pull hard to deform the fluid). This is in accord with the large kinetic energy in the oscillation, which is supplied by the boundary condition (periodically in time, both positive and negative rhythmically). It is important in these estimates that the pressure is nearly hydrostatic, and we estimate it through the x-momentum balance rather than the z-momentum balance.

kH is the aspect ratio (H/L where L is the horizontal scale) of the velocity field. For wide aspect ratio, L >> H, the above inequalities are even more extreme at low frequency. Wide aspect ratios (thin oceans and atmospheres) help the fluid to be hydrostatic and geostrophic.

Math. background: forced ODEs (ordinary differential equations). Bender & Orszag's book (Advanced Mathematical Methods for Scientists and Engineers) gives a short review of ODEs; there are many lengthy textbooks on the subject. In section 1.5 they describe several techniques for forced ODEs: variation of parameters, Green's functions, method of undetermined coefficients. To these I would add Fourier analysis, where we expand F (the forcing term) in sines and cosines, for cases where the homogeneous solutions are sines and cosines.

To quote Bender and Orszag, "Inhomogeneous linear differential equations are only slightly more complicated than homogeneous ones. This is because the difference of any two solutions of Ly = F(x) is a solution of Ly=0. As a result, the general solution of Ly=F(x) is the sum of *any* particular solution of Ly=F(x) and the general solution of Ly=0." [Here Ly means some linear operator like $y_{xx} + Ay$].

This theorem gives us confidence in the procedure where we find a single, forced (particular) solution and then add whatever free (homogeneous) solution we need to satisfy boundary conditions or initial conditions.

Forced oscillator. As a back-ground exercise for **3.** you might recall the forced oscillations of a simple mass-spring oscillator:

$$\eta_{tt} + A^2\eta = F; \quad F = F_0 \sin \omega_0 t$$

with initial conditions: $\eta(t=0)=0$; $\eta_t(t=0)=0$. Do this by the method of 'homogeneous and particular' solutions, that is, assume $\eta = \eta^h + \eta^p$ where η^p is a solution of the forced problem, and η^h is the solution of the homogeneous equation $\eta^h{}_{tt} + A^2\eta^h = 0$ which is used to satisfy the initial conditions. η^h will vary like sin(At) while η^p will vary like sin or cos(ω_0t). Notice the phase of the oscillation compared with the phase of the forcing. Notice the resonance behavior as $\omega_0 \Rightarrow A$. In many textbooks you will read that this problem as two distinct solutions, one varying like $\sin \omega_0 t$ (when $\omega_0 \neq A$) and another varying like $t \sin(\omega_0 t)$ (when $\omega_0 = A$). However if you solve the initial value problem as given here, the solution is general and the limit $\omega_0 \Rightarrow A$ is smooth (and you can understand better the 'textbook' answer).