

Geophysical Fluid Dynamics I P.B. Rhines

Problem Set 3 (revision 1)

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1. Use the energy equation (Gill 8.2) for the one-layer model of a wind-driven flow in a zonal channel to estimate the time for the wind-driven channel flow to develop, without using the time-dependent exact solution. To do this, calculate the KE and APE in the final steady solution, and write down the rate of energy input by the wind-stress (the force $F(y)$).

The ratio [total energy]/[rate of energy input] has the dimensions of time, and is an estimate of the time required to spin up the circulation. Note the general dependence on the energy ratio APE/KE on the scale of the circulation, $L (=l_0^{-1}$ where l_0 is the y-wavenumber) and the Rossby deformation radius.

2. *Forced motion of a single-layer fluid: a local approximation.* In the rotating channel calculation in lectures we threw out some terms in the governing equation for free-surface displacement, η , in order to focus on low-frequency (long time-scale) dynamics. Let us revisit the problem, and think about the response of that fluid shortly after the wind-stress forcing is turned on. At early times the tilt in the fluid surface, $\nabla\eta$, is small. Therefore let us neglect the horizontal pressure gradient and simply solve

$$u_t - fv = F - Ru$$

$$v_t + fu = -Rv$$

In the original problem F is a function of y , but here take it to be a constant (we are looking at the local dynamics in a particular region, before the adjacent regions communicate by signaling through the η -field). Assume that F is switched on at time $t=0$:

$$u(t=0) = 0 = v(t=0)$$

$$F = 0 \quad (t < 0)$$

$$= F_0 \quad (t \geq 0).$$

Solve for $u(t)$, $v(t)$, and particularly look at the oscillating and steady parts of the solution (again, homogeneous and particular solutions), and compare with the theory for the full channel problem (especially the direction the fluid moves in the mean from looking at v/u).

3. *'Stiffening' of the fluid by the Coriolis force..and inertial waves.* The Earth's rotation limits the strength of circulations of ocean and atmosphere. It also 'stiffens' the fluid along the direction of the rotation vector (here vertical). In this problem we will relax the shallow-water, hydrostatic 2-dimensional restriction and look at forced, linear motion of a constant-density rotating fluid without a free surface.

$$u_t - fv = -p_x / \rho$$

$$v_t + fu = -p_y / \rho$$

$$w_t = -p_z / \rho - g$$

$$u_x + v_y + w_z = 0$$

becomes, with no y-variation:

$$u_t - fv = -p_x / \rho$$

$$v_t + fu = 0$$

$$w_t = -p_z / \rho - g$$

$$u_x + w_z = 0$$

We can combine these 4 equations in 4 dependent variables into one equation for any one of u , v , w , or p . The fluid lies between $z = 0$ and $z = -H$. The result is

$$(w_{xx} + w_{zz})_{tt} + f^2 w_{zz} = 0$$

which is an equation for inertial waves.

- solve for a general homogeneous solution in the form of a wave, with frequency ω , and wavevector (k, m) [...that is x-wavenumber is k , z-wavenumber is m].
 $w = \text{Real}(B \exp(ikx + imz - i\omega t))$
- plot or sketch the resulting dispersion relation, $\omega = \text{function of } k \text{ and } m$.
- now force the motion by imposing a vertical motion at the top of the fluid. Let $w(z=0) = A \cos(k_0 x - \omega_0 t)$, and $w(z=-H) = 0$. Solve for $w(x, z, t)$ by writing down the general homogenous solution and then applying both the boundary conditions. It is clear that we will choose $k = k_0$ and $\omega = \omega_0$ so that substituting back in the equation will give us the vertical wavenumber, m . Then the boundary conditions also give the amplitude, B . Note that you can readily satisfy the lower boundary condition by choosing w to vary in z like $\sin(m(z+H))$.
- consider the limit $\omega_0 \ll f$. We will find that: the kinetic energy in the fluid becomes large, the v -velocity becomes much larger than the u -velocity or the w -velocity, and the horizontal velocity v becomes nearly independent of z (the fluid moves almost in columns), the pressure force exerted by the upper boundary condition becomes large, the v -velocity comes close to geostrophic balance and basically, the fluid becomes ‘stiff’. Calculate at least 3 of these results and make a sketch of the velocity field in this limit.

Math. background: forced ODEs (ordinary differential equations). Bender & Orszag’s book (Advanced Mathematical Methods for Scientists and Engineers) gives a short review of ODEs; there are many lengthy textbooks on the subject. In section 1.5 they describe several techniques for forced ODEs: variation of parameters, Green’s functions, method of undetermined coefficients. To these I would add Fourier analysis, where we expand F (the forcing term) in sines and cosines, for cases where the homogeneous solutions are sines and cosines.

To quote Bender and Orszag, “Inhomogeneous linear differential equations are only slightly more complicated than homogeneous ones. This is because the difference of any two solutions of $Ly = F(x)$ is a solution of $Ly=0$. As a result, the general solution of $Ly=F(x)$ is the sum of *any* particular solution of $Ly=F(x)$ and the general solution of $Ly=0$.” [Here Ly means some linear operator like $y_{xx} + Ay$].

This theorem gives us confidence in the procedure where we find a single, forced (particular) solution and then add whatever free (homogeneous) solution we need to satisfy boundary conditions or initial conditions.

Forced oscillator. As a back-ground exercise for **3**, you might recall the forced oscillations of a simple mass-spring oscillator:

$$\eta_{tt} + A^2 \eta = F; \quad F = F_0 \sin \omega_0 t$$

with initial conditions: $\eta(t=0)=0$; $\eta_t(t=0)=0$. Do this by the method of ‘homogeneous and particular’ solutions, that is, assume $\eta = \eta^h + \eta^p$ where η^p is a solution of the forced problem, and η^h is the solution of the homogeneous equation $\eta^h_{tt} + A^2 \eta^h = 0$ which is used to satisfy the initial conditions. η^h will vary like $\sin(At)$ while η^p will vary like \sin or $\cos(\omega_0 t)$. Notice the phase of the oscillation compared with the phase of the forcing. Notice the resonance behavior as $\omega_0 \Rightarrow A$. In many textbooks you will read that this problem as two distinct solutions, one varying like $\sin \omega_0 t$ (when $\omega_0 \neq A$) and another varying like $t \sin(\omega_0 t)$ (when $\omega_0 = A$). However if you solve the initial value problem as given here, the solution is general and the limit $\omega_0 \Rightarrow A$ is smooth (and you can understand better the ‘textbook’ answer).