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Subject: GFD1- PS4.1 and PS4.2

I have added some new notes regarding the bullet 'could this flow have been started from rest?' in PS4.1 and at the end regarding the density field in problem PS4.2. pbr 17 ii 2004 3.30 pm

PS4.1 This afternoon I have had some questions about Problem Set 4. Here's my attempt at answering them. First to correct a typo in my writing the hydrostatic equation ('For reference....'). It should be

$$0 = -\frac{\partial p'}{\partial z} - g\rho'$$

I apologize for the confusion for putting a ρ_0 in the denominator.

The density has the form $\rho_0 \exp(-\gamma z) + \rho'(x, y, z, t)$ in general, though here ρ' is constant in time and in the x-direction. The basic state (with zero velocity) has the background stratification $\rho_0 \exp(-\gamma z)$, and a corresponding hydrostatic pressure $p_0 = (g/\gamma)\exp(-\gamma z)$.

In Problem 1, a velocity field is given and I ask you to find the pressure and density fields assuming geostrophic, hydrostatic balance. With the basic stratification given, we take the density perturbation ρ' to be small compared with the mean density. It also simplifies things if we take the variation of the basic density $\rho_0 \exp(-\gamma z)$ over the depth of the fluid, H , to be smaller compared to ρ_0 . These together are known as the Boussinesq approximation. It is not necessary to make the 2d approximation; we could be thinking of the troposphere with considerable decrease in mean density from top to bottom, but to simplify I recommend assuming $\gamma H \ll 1$

Thus we can just use ρ_0 in the geostrophic horizontal momentum equation, which simplifies things. To sum up all these words:

$$\rho' \ll \rho_0$$
$$\gamma H \ll 1$$

so $\rho \cong \rho_0$ in the momentum equation except where multiplied by g in the buoyancy. The hydrostatic vertical momentum equation sorts out into:

$$\begin{aligned}
p &= p_0(z) + p'(y,z) \\
\partial p_0 / \partial z &= -g\rho_0 \\
\partial p' / \partial z &= -g\rho'
\end{aligned}$$

The pressure perturbation p' comes from one y -integration of the u -velocity in the y -momentum balance. The density perturbation comes from applying the thermal-wind equation which relates the z -derivative of u and the y -derivative of ρ . Or, what is the same, by finding the pressure p' as above and then using the 2d hydrostatic equation above.

In the thermal wind approach, $\partial p / \partial y$ involves just $\rho'(y,z)$ because the basic state ρ_0 is a given function of z only. We are looking at the disturbed, tilted surfaces of constant density that would be level without any velocity field.

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For the bullet 'Could this flow have been started from rest..' it may too difficult for right now; it involves the same sort of logic we have used for several geostrophic adjustment problems. But I will explain it in class and you can omit this one.

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Also, it was pointed out to me that in my lecture notes I have sometimes dropped off a term causing the buoyancy frequency N to have the **wrong units** in the 'parcel' argument. The fundamental definition of buoyancy frequency squared, N^2 , for an incompressible fluid, is $-(g/\rho)(dp/dz)$. With compressibility it has a correction involving the sound speed squared. Because g has dimensions length/time², N^2 has dimensions 1/T² as it should. I wrote at one point

$N^2 = -g\alpha(T_{\text{parcel}} - T_{\text{surroundings}})$ or $N^2 = -g(\delta\rho)$ which have the wrong units. What I meant to say is:

If we call

$\delta\rho = [\rho \text{ of the parcel}] - [\rho \text{ of the surrounding fluid}]$
the buoyant force per unit volume, the vertical force on a displaced fluid parcel is

$$-g \delta\rho.$$

From " $F = ma$ ", this restoring force causes a vertical acceleration equal to the 'reduced gravity acceleration'

$$-g (\delta\rho)/\rho \quad (\text{whose dimensions are length/time}^2)$$

because " ma " is $\rho \, dw/dt$ per unit volume. In terms of temperature, T , this involves

$$g \alpha \delta T,$$

with α being the expansion coefficient,

$$\alpha \equiv - (1/\rho) \partial\rho / \partial T|_p \text{ taken at constant pressure.}$$

In the corresponding oscillator equation

$$\begin{aligned}
d^2(\delta Z) / dt^2 &= -g \delta\rho / \rho \\
&= - (g/\rho) (dp/dz + g\rho/c^2) \delta z
\end{aligned}$$

the square of the frequency is

$$\begin{aligned}
& -g (\delta\rho)/(\rho \delta z) \\
& = -(g/\rho (d\rho/dz + g\rho/c^2))
\end{aligned}$$

which we write as above, and also have included the correction of $\delta\rho$ for the compressibility effect, using the squared speed of sound, c^2 , defined by the derivative of pressure with respect to density at constant entropy

$c^2 = (\partial p/\partial\rho)|_\eta$ (for a perfect gas, $c^2 = \gamma RT$ where γ here (not to be confused with the γ above) is the ratio of specific heat capacities, $\gamma = C_p/C_v$).

PS4.2

I had a question this morning about problem 2 of Problem Set 4.

***Basically, how do we figure out how the density field evolves in time? and...how do we calculate the full density field, $\rho(x,z,t)$?**

The answer is actually implied by the last equation on the page, involving the x and t derivative of u.

We argued that the problem is uniform with respect to x, except for the linear variation of the initial density, $\rho = \rho_0 + Ax$. Ignoring any end-wall effects, having $\partial u/\partial x = 0$ implies $\partial w/\partial z = 0$ from mass conservation. With $w = 0$ on top and bottom boundaries, this guarantees that $w = 0$ everywhere. Now look at the density equation (the second half of mass-conservation). With $w=0$, it is

$$\partial\rho/\partial t = - u \partial\rho/\partial x$$

Differentiate once more with respect to x:

$$\partial^2\rho/\partial t\partial x = - u \partial^2\rho/\partial x^2$$

Now at initial time, $t=0$, the righthand side is zero (since $\rho = \rho_0 + Ax$). The lefthand side is the time rate of change of the horizontal density gradient.

So, the partial $\partial\rho/\partial x = A$ for all time...the horizontal density gradient does not change with time.

This is the basis for writing the final expression in the equation for the x and t derivative of u (the term gA/ρ_0).

After solving for partial $\partial v/\partial z$, consider its time average value (the particular solution). You will find it is in thermal wind (geostrophic+hydrostatic) balance with the density field.

Make sketches. A consequence of the above result is that the fluid moves in 'rigid sheets' horizontally (though that motion is a function of z). Lines of constant density in the x-z plane are initially vertical but as they 'slump' they remain straight and parallel with one another. So, calculating $\partial\rho/\partial z$ is not difficult, and then you have N.

To actually calculate $\partial\rho/\partial z$ you can use a geometric argument based on the tilting lines of constant density, or more methodically do the following: we know that

$$v_t + fu = 0$$

$$\rho_t = -u\rho_x$$

differentiate these to give

$$v_{zt} = -fu_z$$

$$\rho_{zt} = -u_z\rho_x - u\rho_{zx} = -u_z\rho_x = -Au_z \quad \text{since } \rho_{zx} = A_z = 0.$$

now, combine to give an equation for ρ_{zt} in terms of v_{zt} and solve it for ρ_z and then N^2 .

Your sketches should look like a 'Venetian blind' where each line of constant density stays straight while it pivots about a fixed point halfway between top and bottom boundaries, and all the different constant-density lines are parallel.

This problem uses the Boussinesq approximation also (you see ρ_0 in the horizontal pressure gradient terms). I will try to hand out notes on this approximation because it is subtle.

p.s. To answer another question on problem 1, calculating the free-surface displacement from the pressure, just use the mean density ρ_0 and not the full density, again in the spirit of the Boussinesq approximation.