

## Geophysical Fluid Dynamics I P.B. Rhines SOLUTIONS vsn 2

### Problem Set 4

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1. An east-west flow exists with the form

$$u = U_0 \cos(l_0 y) \exp(\kappa z)$$

The domain is  $-H < z < 0$ , and the surface  $z = 0$  is rigid.

It is a geostrophic, hydrostatic flow, with no friction effects. There is a density stratification, with  $\rho = \rho_0 \exp(-\gamma z) + \rho'(y, z)$  where  $\rho'/\rho_0 \exp(\gamma H)$  is small;  $\rho'$  is due to the presence of the velocity field;  $\gamma H \ll 1$  (note that  $1/\gamma$  is the scale height of the density field). Consider the fluid to be incompressible (more like water than air) so that we are concerned with potential temperature or potential density.

• Find the pressure  $p(y, z)$ , density perturbation, and buoyancy frequency,  $N$ . (Assume that the pressure gradient term is  $-\nabla p/\rho_0$  rather than  $-\nabla p/\rho$ , since  $\rho_0$  is very large).

Let  $p = p_0(z) + p'(y, z)$  since there is no  $x$ -variation in  $u$ . Hydrostatic balance is

$$\frac{\partial p_0}{\partial z} = -\rho_0 \exp(-\gamma z) \quad \Rightarrow \quad p_0 = (\rho_0 / \gamma) \exp(-\gamma z)$$

$p'$  is found from geostrophic balance:  $p' = -\rho_0 f \int u dy = -(\rho_0 f U_0 / l_0) \sin l_0 y e^{\kappa z}$

$\rho'$  is found from hydrostatic balance with  $p'$  (or from integrating thermal wind equation):

$$\rho' = -\frac{\partial p'}{\partial z} / g = +(\kappa \rho_0 f U_0 / g l_0) \sin l_0 y e^{\kappa z}$$

The buoyancy frequency squared is

$$N^2 = -(g / \rho) \frac{\partial \rho}{\partial z} = -\frac{g}{\rho_0 \exp(-\gamma z) + \rho'} (\gamma \rho_0 e^{-\gamma z} + \frac{\partial \rho'}{\partial z})$$

where  $\rho'$  is as above or,

$$N^2 \cong (g / \rho_0) \gamma \rho_0 e^{-\gamma z} \cong g \gamma$$

the final simple result using the Boussinesq approximation. Note that the units check ok, dimensions of  $N^2$  are  $1/\text{time}^2$ .

• Find the density,  $\rho'(y, z)$ . Sketch the surfaces of constant density and constant pressure in the  $y$ - $z$  plane (which of these have steeper slope?). If  $\rho'$  is a function of temperature only, where are the warm and cold regions in the flow, in relation to high- and low pressure regions? Compare the steepness of the slopes of surfaces of constant density with the surfaces of constant pressure...you can use for the constant pressure surface

$$slope = \left. \frac{\partial z}{\partial y} \right|_p = \frac{\partial p / \partial y}{\partial p / \partial z}$$

and similarly for constant density surfaces.

$\rho'$  is given above. Sketch: The velocity  $u$  is a series of alternating zonal jets, which are stronger at the top of the fluid, weaker at the bottom. Each velocity core corresponds to a maximum tilt of the density surfaces, and the valleys of the density field (the warm cores) occur south of the maximum westerlies, where the vertical vorticity is negative. Cold cores occur north of the maximum westerlies, with cyclonic vorticity.

•If the surface  $z=0$  were replaced by a free surface where the pressure is equal to atmospheric pressure, what would be the elevation of the surface,  $\eta(y)$ ? Sketch it. [We assume  $g$  is sufficiently large that the free surface at  $z=\eta$  is very close to  $z=0$ .] How do the combined effects of  $\eta$  and  $\rho'$  yield the calculated pressure field at  $z = -H$ ?

Use the Boussinesq approximation, so that  $p' = \rho_0 g \eta$  where  $\eta$  is the free-surface displacement, hence  $\eta =$  above expression for  $p$  divided by  $\rho_0 g$ .

•Could this flow have been started from a rest state ( $u=0, v=0, w=0$ ) with some simple initial density ( $\rho'$ ) and free surface ( $\eta$ ) profiles?

This question relates to geostrophic adjustment...the initial-value problems we have been looking at. I have asked the students to skip this one. We will discuss it in class.

•Describe the potential energy and kinetic energy of this flow, and the ratio APE/KE, relating them to the Rossby deformation radius,  $L_\rho$ , which for a continuously stratified fluid, and a motion with vertical scale  $1/\kappa$ , is

$$L_\rho = N/\kappa f.$$

See the energy equation in Gill p. 80 (Sec 4.7). For a stratified fluid we often use an approximation to the APE (available potential energy) given in Gill p. 140 (sec 6.7), which is  $\iiint \frac{1}{2} \frac{g^2 \rho'^2}{\rho_0 N^2} dx dy dz$ . Read also

Gill Sec. 7.8.

Using the above results we have

$$APE = \frac{1}{2} \rho_0 \frac{\kappa^2 f^2}{N^2 l_0^2} U_0^2 \sin^2(l_0 y) \exp(2\kappa z)$$

$$KE = \frac{1}{2} \rho_0 U_0^2 \cos^2(l_0 y) \exp(2\kappa z)$$

Take the mean values of the  $\sin^2, \cos^2$ , then the ratio is

$$APE / KE = \left( \frac{\kappa f}{l_0 N} \right)^2$$

which can be written  $(fL/NH^*)^2$  or  $(L/L_\rho)^2$  based on the horizontal length scale  $L = 1/l_0$  and vertical length scale of the velocity,  $H^* = 1/\kappa$ .

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**For reference,  $f$ -plane MOM equation:** 
$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + f \hat{z} \times \vec{u} = -\frac{1}{\rho} \nabla p - g \hat{z} + \nu \nabla^2 \vec{u}$$

where  $f = 2\vec{\Omega} \cdot \hat{z} \sin(\text{mean latitude})$ ,  $\hat{z}$  being a vertical unit vector. (The exact equation has a Coriolis term  $2\vec{\Omega} \times \vec{u}$  and  $-\nabla\Phi$  instead of  $-g\hat{z}$ , where  $\Phi$  is the geopotential).

*Geostrophic and hydrostatic MOM equations, with small density variations:*

$$-fv = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}; fu = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y}; \quad 0 = -\frac{\partial p'}{\partial z} - \rho' g \quad (2) \quad \text{where } \rho = \rho_0 \exp(-\gamma z) + \rho'(x,y,z),$$

$\rho_0$  being a constant and the pressure is the mean hydrostatic pressure due to  $\rho_0 \exp(-\gamma z)$  plus the motion-induced pressure,  $p'(x,y,z)$ . These equations use the *Boussinesq approximation*, which is to say that  $\rho'/\rho_0$  is in some sense small, so that we can just use the mean density  $\rho_0$  in terms like  $1/\rho_0 \partial p/\partial x$  yet we use the full density when expressing the buoyancy,  $g\rho$ .

Because the fluid is incompressible,  $N^2 = -(g/\rho)d\rho/dz$ .

*Thermal wind equation:* eliminate  $p$  from equations (2) by cross-differentiation.

## 2. Geostrophic Adjustment with stratification.

Consider the development of flow in a layer of fluid, confined between upper and lower boundaries at  $z=0$  and  $z=-H$ : the fluid at time  $t=0$  has *horizontal stratification*,

$$\rho = \rho_0 + Ax$$

That is, the constant-density surfaces are vertical!. This situation might arise, for example, if a layer of fluid were mixed by turbulence while being cooled on one end and heated on the other. If we assume this layer of fluid is uniform in  $x$  and any vertical boundaries are very far away, the equations are

*mom:*

$$u_t - fv = -p_x / \rho_0$$

$$v_t + fu = 0$$

$$p_z = -g\rho$$

*mass:*

$$\rho_t + u\rho_x = -w\rho_z$$

$$u_x + w_z = 0$$

Here we have no variation in the  $y$ -direction. At  $t=0$  the horizontal pressure gradient  $p_x$  is independent of  $x$  and  $y$  so the  $u$  and  $v$  velocities will be independent of  $x$  and  $y$  for all time (since they are time integrals of pressure gradient). Thus using MASS conservation we find

$$w=0$$

for all time: the motion moves purely horizontally. Now in the real world, the end walls (neglected here) will cause vertical motion and this is what is involved in release of APE (potential energy), but we can say that that happens 'later'.

Show that the x-mom equation becomes, after differentiation in z:

$$u_{zt} - fv_z = -p_{zx} / \rho_0 = gA / \rho_0 \quad (*)$$

Thus find an equation for  $v_z$  (...  $\partial v / \partial z$  that is) which will be a forced o.d.e. with respect to time. **Solve for the time dependent solution for  $v_z$  and show that it is the sum of a geostrophic flow in thermal wind balance (after time averaging) plus inertial oscillations.**

Calculate the mean value of  $N^2$  that results from this slumping density field, and sketch the velocity and surfaces of constant density.

**The first part of equation (\*) follows** from the z-derivative of the x-mom equation. The second part follows from the fact that the horizontal density gradient,  $\rho_x$  is independent of time. To see this, look at the MASS conservation equations,

$$\rho_t + u\rho_x = -w\rho_z, \quad u_x + w_z = 0$$

With the problem uniform in the x-direction except for the initial linear density gradient in x, we expect  $\partial u / \partial x = 0$ . Hence  $\partial w / \partial z = 0$  and with w equal to zero on top and bottom this means  $w=0$  everywhere. Now the 1<sup>st</sup> of the MASS conservation equations can be differentiated in x to show  $\rho_{xt} = -u\rho_{xx} = 0$  at the initial time. So,  $\rho_x$  does not change in time, and it equals A for all time.

Now use the other MOM equation to turn (\*) into a single equation for  $v_z$ ,

$$v_{ztt} + f^2 v_z = -gfA/\rho_0$$

which is a forced o.d.e. with constant coefficients (2d order to solve for  $v_z$ ).

The solution is (homogeneous plus particular, inertial oscillations plus mean flow)

$$v_z = A \sin ft + B \cos ft - gA/f\rho_0.$$

with initial condition  $v_z = 0$  at  $t=0$ . The full solution is

$$v_z = (gA/f\rho_0)(\cos ft - 1).$$

This is independent of z. Integrate once in z:

$$v = -(z - \frac{1}{2} H)(gA/f\rho_0)(1 - \cos ft).$$

where we chose the constant of integration to reflect the symmetry of the problem..the upper fluid goes right (toward positive x) while the lower half goes left.

This solution describes a flow that in the time-average is in geostrophic balance as can be seen from the basic equations above, or by direct calculation. The mean geostrophic thermal wind velocity,  $v = -(z - \frac{1}{2} H)(gA/f\rho_0)$  is a linearly sheared flow in the y-direction, normal to the gradient of density.

Now solve for the full u and  $\rho$  fields. Use:

$$v_t + fu = 0 \quad \text{which says simply } v = f(X-X_0), \text{ based on the x-}$$

displacement of a fluid particle  $X(t) = \int u dt$  from its origin  $X_0$ .

$$\rho_t = -u\rho_x = -Au$$

to give

$$u = (z - \frac{1}{2} H)(gA/f\rho_0)\sin ft$$

$$X = (z - \frac{1}{2} H)(gA/f^2\rho_0)(1 - \cos ft)$$

$$\rho = \rho_0 + Ax - (z - \frac{1}{2}H)(gA^2/f^2\rho_0)(1 - \cos ft)$$

which is also written  $\rho = \rho_0 + A(X - X_0)$  (showing conservation of density following the fluid). Note that  $X$  has a non-zero mean value even though  $u$  does not.

This gives immediately the mean density structure. Note, we have assumed that since the problem is symmetric vertically about the mid-depth that the density surfaces pivot about that  $z$ -level. Distant boundary conditions could cause them to pivot about some other  $z$ -level (top or bottom for example), but the results are all similar.

Notes:

- time average is thermal wind balanced,  $f v_z = -g \rho_x / \rho_0$  ( $= -gA / \rho_0$ )

- buoyancy frequency is given by

$$N^2 = (gA / \rho_0 f)^2 (1 - \cos ft)$$

- the mean slope of the isopycnal surfaces is

$$\begin{aligned} \frac{1}{2} H / [\text{mean of } X(z=0)] \\ &= \rho_0 f^2 / gA \\ &= f / N \end{aligned}$$

which is the classic result, slope of density surfaces in geostrophic flows  $\sim f/N$  if their horizontal scale is the Rossby deformation radius,  $NH/f$ . Here the fluid builds its own vertical stratification.

- the inertial oscillation brings the fluid repeatedly back to its initial position, but we imagine these would decay with friction or radiate away.

- for future reference, the Ertel-Rossby potential vorticity here is zero (which is clear from the initial conditions, where  $\nabla \rho$

- It is interesting that the ratio

$$Ri = N^2 / [(\partial v / \partial z)^2 + (\partial u / \partial z)^2]$$

is just  $1/[2(1 - \cos ft)]$  as a function of time, or 1 based on the time-averaged geostrophic solution. This ratio  $Ri$  is called the *Richardson number*, and it is a measure of the tendency of a stratified shear flow to go unstable and break up into eddies. If  $Ri < \frac{1}{4}$  unstable billows can develop. Here the flow oscillates between  $Ri = \infty$  and  $Ri = \frac{1}{4}$  each inertial period, and hence is very close to instability of this kind.

3. Write one page (single-spaced) discussion of one of the laboratory experiments you have seen, one with Coriolis effects.