Geophysical Fluid Dynamics I   P.B. Rhines
Problem Set 4
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1. An east-west flow exists with the form

$$u = U_0 \cos(l_0 y)\exp(\kappa z)$$

The domain is $-H < z < 0$, and the surface $z = 0$ is rigid.

It is a geostrophic, hydrostatic flow, with no friction effects. There is a density stratification, with $\rho = \rho_0 \exp(-\gamma z) + \rho'(y,z)$ where $\rho'/\rho_0 \exp(\gamma H)$ is small; ρ' is due to the presence of the velocity field; $\gamma H \ll 1$ (note that $1/\gamma$ is the scale height of the density field). Consider the fluid to be incompressible (more like water than air) so that we are on concerned with potential temperature or potential density.

•Find the pressure p(y,z), density perturbation, and buoyancy frequency, N. (Assume that the pressure gradient term is $-\nabla p/\rho_0$ rather than $-\nabla p/\rho$, since ρ_0 is very large).

•Find the density, $\rho'(y,z)$. Sketch the surfaces of constant density and constant pressure in the y-z plane (which of these have steeper slope?). If ρ' is a function of temperature only, where are the warm and cold regions in the flow, in relation to high- and low pressure regions? Compare the steepness of the slopes of surfaces of constant density with the surfaces of constant pressure…you can use for the constant pressure surfaces

$$slope = \left.\frac{\partial z}{\partial y}\right|_p = \frac{\partial p/\partial y}{\partial p/\partial z}$$

and similarly for constant density surfaces.

•If the surface z=0 were replaced by a free surface where the pressure is equal to atmospheric pressure, what would be the elevation of the surface, $\eta(y)$? Sketch it. [We assume g is sufficiently large that the free surface at $z=\eta$ is very close to z=0.] How do the combined effects of η and ρ' yield the calculated pressure field at z = -H?

•Could this flow have been started from a rest state (u=0, v=0, w=0) with some simple initial density (ρ') and free surface (η) profiles?

•Describe the potential energy and kinetic energy of this flow, and the ratio APE/KE, relating them to the Rossby deformation radius, L_ρ, which for a continuously stratified fluid, and a motion with vertical scale $1/\kappa$, is

$$L_\rho = N/\kappa f .$$

See the energy equation in Gill p. 80 (Sec 4.7). For a stratified fluid we often use an approximation to the APE (available potential energy) given in Gill p. 140 (sec 6.7), which is $\iiint \tfrac{1}{2}\frac{g^2 \rho'^2}{\rho_0 N^2} dx\,dy\,dz$. Read also Gill Sec. 7.8.

For reference, *f-plane MOM equation*: $$\frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u} + f\hat{z} \times \vec{u} = -\frac{1}{\rho}\nabla p - g\hat{z} + \nu \nabla^2 \vec{u}$$

where $f = 2\vec{\Omega} \bullet \hat{z}\ \sin(mean\ latitude)$, $\hat{z}$ being a vertical unit vector. (The exact equation has a Coriolis term $2\vec{\Omega} \times \vec{u}$ and $-\nabla\Phi$ instead of $-g\hat{z}$, where Φ is the geopotential).

Geostrophic and hydrostatic MOM equations, with small density variations:

$$-fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x};\ fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}; 0 = -\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g \quad (2)$$

where $\rho = \rho_0 \exp(-\gamma z) + \rho'(x,y,z)$, ρ_0 being a constant and the pressure is the mean hydrostatic pressure due to $\rho_0 \exp(-\gamma z)$ plus the motion-induced pressure, $p'(x,y,z)$. These equations use the *Boussinesq approximation*, which is to say that ρ'/ρ_0 is in some sense small, so that we can just use the mean density ρ_0 in terms like $1/\rho_0\ \partial p/\partial x$ yet we use the full density when expressing the buoyancy, $g\rho$.

Because the fluid is incompressible, $N^2 = -(g/\rho)d\rho/dz$.

Thermal wind equation: eliminate p from equations (2) by cross-differentiation.

2. Geostrophic Adjustment with stratification.

Consider the development of flow in a layer of fluid, confined between upper and lower boundaries at z=0 and z=-H: the fluid at time t=0 has *horizontal* stratification,

$$\rho = \rho_0 + Ax$$

That is, the constant-density surfaces are vertical!. This situation might arise, for example, if a layer of fluid were mixed by turbulence while being cooled on one end and heated on the other. If we assume this layer of fluid is uniform in x and any vertical boundaries are very far away, the equations are

$$\begin{aligned} &mom: \\ &u_t - fv = -p_x/\rho_0 \\ &v_t + fu = 0 \\ &p_z = -g\rho \\ &mass: \\ &\rho_t + u\rho_x = -w\rho_z \\ &u_x + w_z = 0 \end{aligned}$$

Here we have no variation in the y-direction. At t=0 the horizontal pressure gradient p_x is independent of x and y so the u and v velocities will be independent of x and y for all time (since they are time integrals of pressure gradient). Thus using MASS conservation we find

$$w=0$$

for all time: the motion moves purely horizontally. Now in the real world, the end walls (neglected here) will cause vertical motion and this is what is involved in release of APE (potential energy), but we can say that that happens 'later'.

Show that the x-mom equation becomes, after differentiation in z:

$$u_{zt} - fv_z = -p_{zx}/\rho_0 = gA/\rho_0$$

Thus find an equation for v_z (…$\partial v/\partial z$ that is) which will be a forced o.d.e. with respect to time. **Solve for the time dependent solution for v_z and show that it is the sum of a geostrophic flow in thermal wind balance (after time averaging) plus inertial oscillations.**

Calculate the mean value of N^2 that results from this slumping density field, and sketch the velocity and surfaces of constant density.

3. Write one page (single-spaced) discussion of one of the laboratory experiments you have seen, one with Coriolis effects.