

Geophysical Fluid Dynamics I P.B. Rhines .. typo corrected vsn 1

Problem Set 5

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1. Consider flow of a uniform density fluid ($\rho = \rho_0$) over a mountain range, where the fluid depth, h , is

$$\begin{aligned} h &= H + \tilde{h}(x) \\ &= H + d \sin(k_0 x) \end{aligned}$$

Assume that the fluid flows from west to east (toward positive x), and has uniform potential vorticity (PV). This would be the case, for example if far upstream the depth were uniform (no mountains) and the flow were uniform (constant u -velocity). Use the potential vorticity equation in the form

$$\begin{aligned} Dq / Dt &= 0; \\ q &= (f + \zeta) / h \approx f / H + \zeta / H - f\tilde{h} / H^2 \end{aligned}$$

for steady flow this becomes

$$\tilde{u} \bullet \nabla q = 0;$$

if we linearize the problem for small topography ($d/H \ll 1$) then the mean zonal flow is much bigger than the modification of the flow due to the topography: $u = U + u'(x)$, $v = v'$ with $u' \ll U$, and the equation becomes

$$U \frac{\partial q}{\partial x} = 0$$

which says that $q = \text{constant}$ or at most a function of y . We have shown that the vertical vorticity is the Laplacian of the geostrophic streamfunction (ψ), or of geostrophic pressure (p). Thus the equation becomes

$$\nabla^2 \psi = f\tilde{h} / H + \text{const. or } m(y)$$

Because the upstream flow has uniform PV we don't need the function of y .

- Thus solve for ψ with the above topography. Sketch the streamlines. Note that positive \tilde{h} corresponds to a valley (negative height mountain).
- Find the solution for any topography $\tilde{h}(x)$, assuming the upstream flow is uniform and the PV is uniform. Taking an example as an isolated mountain ridge, show that v , the y -velocity component ('along' the ridge) experiences a change proportional to the *volume* of the ridge, $\int \tilde{h} dx$. An example of such a ridge would be $\tilde{h} = d \exp(-x^2 / a^2)$. Thus the ridge produces a *vortex sheet* in the fluid, shifting the direction of the zonal flow.

2. Thermal wind. In the horizontal – x -direction - vorticity equation, use scale analysis to compare the approximate size of the neglected terms: production of relative horizontal vorticity, say $D\zeta^{(x)}/Dt$, and the term $(\partial p / \partial y)(\partial \rho / \partial z)$ compared to the main thermal wind terms, $(\partial p / \partial z)(\partial \rho / \partial y)$ and $f \partial u / \partial z$. Here the x -component of vorticity is $\zeta^{(x)} = \partial w / \partial y - \partial v / \partial z$. [Note that the term $D\zeta^{(x)}/Dt$ the generation of x -vorticity, is important in geostrophic adjustment: in the PS4.2 homework buoyancy twisting makes the fluid slump, with strong horizontal vorticity along the same axis as the twisting, but then Coriolis forces 'arrest' the slumping, negate that component of horizontal vorticity, and settle in with the other component of horizontal vorticity in thermal-wind balance.]

3. Fourier analysis of flow fields. The procedure above basically involves solving for velocity, pressure, streamfunction when we know the vorticity, $\zeta = \nabla^2 \psi$. This involves a *Poisson equation*

$$\nabla^2 \psi = F(x, y)$$

which is an inhomogeneous, 'forced' Laplace equation. Let us think about this using some ideas of Fourier analysis.

Suppose we write ψ and F as a Fourier series, a sum of sines and cosines, each with an amplitude factor:

$$\begin{aligned} \psi &= \text{Re } al \left[\sum_n \hat{\psi}_n \exp(ik_n x + il_n y) \right] \\ F &= \text{Re } al \left[\sum_n \hat{F}_n \exp(ik_n x + il_n y) \right] \quad k_n = 2\pi n / L, \quad l_n = 2\pi n / L \end{aligned} \tag{1}$$

These sine-waves have wavenumbers which are a multiple of the constant $2\pi/L$, hence ψ and F are periodic in x , over a distance L .

- Show that Poisson's equation can be written in terms of the Fourier coefficients as

$$-(k_n^2 + l_n^2)\hat{\psi}_n = \hat{F}_n \quad (2)$$

by substitution, and equating coefficients of each $\exp(i\dots)$ to zero.

Thus $\hat{\psi}$ is 'amplified' at small wavenumber and reduced in strength at large wavenumber, compared with \hat{F} . We can say ψ is a *low-pass filtered image of F*. This is just what one does in image-analysis to soften or sharpen an image. In engineering, there is a maxim, 'Integration smoothes, differentiation roughens', which is saying the same thing.

The Laplace operator ∇^2 of the height of a surface is proportional to its *curvature* (for small amplitude). The equation for deflection, ψ , of a string (one-dimensional) or elastic membrane (two-dimensional) is the above Poisson equation where $F(x,y)$ represents a distribution of weights or lifting forces. This physical analog helps in visualizing solutions.

- For our fluid case, the kinetic energy of the fluid is $\frac{1}{2} \rho_0 |\nabla\psi|^2$. Show that the contribution to the kinetic energy from a given Fourier component, say $KE(k_n, l_n)$, is $\frac{1}{4} \rho_0 (k_n^2 + l_n^2) |\hat{\psi}_n|^2$ (where we need to be careful about taking the real part in the definition (1)). The idea is that the sum of KE over all wavenumbers equals the integral of the kinetic energy in space,

$$\frac{1}{4} \rho_0 \sum_n (k_n^2 + l_n^2) |\hat{\psi}_n|^2 = \iint \frac{1}{2} \rho_0 |\nabla\psi|^2 dx dy$$

{This is in fact known as Parseval's Theorem. Note, $\text{Real}(m) = \frac{1}{2} (m+m^*)$ where m^* is the complex conjugate of m }.

KE, is known as the *kinetic energy spectrum* or the *velocity spectrum*, and it tells us the distribution of energy among wavenumbers. In the atmosphere, the kinetic energy spectrum peaks near zonal-wavenumbers 3 to 5. (that is, 3 to 5 sine-waves around a latitude circle). Thus we see that *the wavenumber spectrum of pressure or of ψ (one is proportional to the other)*, is a 'low-pass filtered' version of the *velocity spectrum (or kinetic energy spectrum), KE*. That is: if we define $P = \frac{1}{4} (\rho_0 f)^2 |\hat{\psi}_n|^2$ then $KE = (1/\rho_0 f^2)(k_n^2 + l_n^2)P$ and similar to the result above, the spatial field of geostrophic pressure is a smoothed field compared with spatial maps of the u and v velocity components.

- Find the relation between the wavenumber spectrum of vertical vorticity, $\nabla^2\psi$, and the spectra of kinetic energy, KE, and pressure, P.
- Suppose that the fluid flow is now stratified and geostrophic, with constant buoyancy frequency, N . Include a vertical dependence, and vertical wavenumber (periodic in the depth H of the fluid region),

$$\psi = \text{Real} \left[\sum_n \hat{\psi}_n \exp(ik_n x + il_n y + im_n z) \right] \quad m_n = n2\pi / H$$

We want to know the horizontal wavenumber content of the perturbation density, $\rho'(x,y,z)$. Show that it will be similar to pressure if the vertical variation has just a single wavenumber, yet different from the pressure if there is a rich vertical distribution of wavenumbers.

Construct the spectrum APE (of available potential energy), from its definition given in Gill p. 140 (sec 6.7),

which is $\frac{1}{2} \frac{g^2 \rho'^2}{\rho_0 N^2}$ per unit volume, writing it in terms of the pressure spectrum, and show how the spectral ratio APE/KE

depends on your assumption about the relation between vertical and horizontal wavenumbers. This is a more exact way, compared to scale analysis, of looking at energetics.

- Illustrate some of these results by constructing a 2-dimensional flow field from an assumed 'random' distribution of Fourier components and making plots of $\psi(x,y)$, $\zeta(x,y)$, $u(x,y)$. If you have access to Matlab, or a similar calculating/plotting engine, use many Fourier components (many wavenumbers) and plot contour maps of ψ , p , u , v , and ζ . But if not just try two, say $k = k_0$ and $k = 4k_0$. How would you expect the spectrum of *particle displacement* $\int \vec{u} dt$ to behave?