GFD-1 Winter 2004 Take-home Quiz #2
out: Thursday 4 March
back: Friday 5 March before 3.00pm to Dan in At.Sci. 620 or PBR in Ocean Sciences 319
Books and notes may be used, but please do not communicate with anyone about this quiz.

1. Consider a rotating, stratified fluid driven by heating and cooling: a model of the meridional overturning circulation. The fluid is in a rectangular channel running east and west, along the x-axis. The channel walls are at y= 0, L and z= 0,H. It is cooled in the north (large y) and warmed in the south (small y) by radiative forcing which acts on the interior region of fluid (see below). The forcing is independent of x; *thus assume that no part of the solution varies in x.* The forcing of the density equation has the form

$$\wp = C\sin(2\pi y/L)\sin(\pi z/H)$$

(density varies inversely to temperature so take C to be a negative number).

(a) Describe in words what you expect to happen (and **draw a sketch**). The fluid is uniformly stratified at the start, and we will assume that the changes in stratification are small over the life-time of the experiment.

Now, we want to solve the quasi-geostrophic equations to learn more. We have used the potential vorticity (PV) equation to study zonal (east-west) flow over mountains, in which case the PV was conserved following a fluid parcel. Here we will add a forcing term F on the right-hand side of the PV equation,

$$\frac{D}{Dt}q \equiv q_t - \psi_y q_x + \psi_x q_y = F$$

$$q \equiv \psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz}$$

This is the quasi-geostrophic form of PV as in lectures and the hand-out notes. If we ignore the advective part of the total time derivative this becomes

$$\boxed{\frac{\partial}{\partial t}(\psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz}) = F} \qquad (1)$$

where N is assumed to be a constant, independent of z. ψ is the geostrophic streamfunction for the (u,v) horizontal velocity; ψ is proportional to geostrophic pressure. *F is the forcing, which here involves heating and cooling.*

(b) We first need to connect the heating/cooling with the forcing of PV, the term F. The density field is

$$\rho = \rho_0(z) + \rho'(y,z,t)$$

where ρ_0 gives the basic, uniform stratification (with buoyancy frequency N, a constant).
The buoyancy forcing (the heating and cooling) appears in the density equation as

$$\frac{D\rho}{Dt} \cong \frac{\partial\rho'}{\partial t} + w\frac{d\rho_0}{dz} = \wp(y,z) \qquad \rho = \rho_0(z) + \rho'(y,z,t) \qquad (2)$$

$$\textit{where}\ \ \wp = C\sin(2\pi y/L)\sin(\pi z/H)$$

where we ignore the advection of ρ'. Note C < 0 gives warming in the south and cooling in the north. Using this equation
show (following our derivation of the PV equation from the vertical vorticity equation) **that the PV equation** is forced by a term

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$$F = A\frac{\partial(B\wp)}{\partial z} \qquad (3)$$

What are A and B?

(c) Equation (1) can be regarded as an equation for $\frac{\partial\psi}{\partial t}$.

Solve (1) with the forcing (3), for $\frac{\partial\psi}{\partial t}$**,** the rate of change of ψ forced by this particular form for F(y,z). Assume the solution to be independent of x. Separation of variables works, assuming a form of solution with just the same (y,z) structure as the forcing term. The boundary conditions, $\partial\psi/\partial z=0$ on z = 0, H; v = 0 on y = 0, L will be automatically satisfied in this solution.

The circulation increases linearly with time*.

Write down and sketch the zonal velocity, u, and density perturbation, ρ' as a function of y and z some value of the time.

Show how thermal-wind balance applies here.

Solve for and sketch the (v,w) velocity field in the meridional (y,z) plane. The vertical velocity w appears in equation (2), and the north-south velocity v can then be found using conservation of mass (note, v is not geostrophic, since there is no variation of pressure in the x-direction, and thus it is not equal to ψ_x). For small density differences mass conservation gives approximately

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Sketch the constant density surfaces $\rho_0 + \rho'$ in the (y,z) plane.

What is the pattern of vortex stretching, and what does it produce?

(d) Using scale analysis,

describe how the ratio of available potential energy divided by kinetic energy, APE/KE depends on the scales L and H. Kinetic energy is $\frac{1}{2}\rho_0(\psi_x{}^2 + \psi_y{}^2)$ and available potential energy is $\frac{1}{2}\rho_0\frac{f^2}{N^2}(\psi_z)^2$

(e) If the problem is modified by including a north-south boundary, as we have in the oceans, **describe** how you think the Coriolis force on the north-south velocity, v, can now be balanced.

***** The meridional overturning circulation cannot become steady, time-independent unless we add damping terms that destroy momentum and density differences. Understanding these damping terms is an active research area in both ocean and atmosphere.