**Geophysical Fluid Dynamics – I P.B. Rhines 29 ii 2004**

**Geostrophic Flow of a Density-Stratified Fluid** ***Revision 2***

## 1. Fluid at Rest: vertical structure, hydrostatic balance, buoyancy frequency

The thin sheets of fluid which are the atmosphere and oceans have a hydrostatic vertical momentum balance, if the aspect ratio of the motion, is small, $(H/L)^2 << 1$. Then the pressure at a given point is equal to the weight of the fluid overhead, per unit area, plus any pressure existing at the 'top' of the fluid. This would not be the case in a thick atmosphere, where one has to use spherical equations rather than Cartesian ones, as Isaac Newton once proved. The unit of pressure is the Pascal, Pa, or Newton meter$^{-2}$, with units $M\tilde{T}^{-2}L^{-1}$. ($(M, L, \tilde{T}) =$ mass,length,time units.) The atmospheric pressure at the ground is approximately $10^5$ Pa, known as 1 atmosphere, 1 bar, 1000 millibars, 760mm Hg (the height of a mercury barometer column with a vacuum at the top), or about 10 m of water. Pressure in the ocean is measured in decibars ($10^{-1}$ bar); 1 decibar is $10^4$ Pa, close to 1 m of water height because the density of ocean water is about 1028 kg m$^{-2}$ and g is 9.8 m sec$^{-2}$. At the bottom of a 5 km deep ocean the pressure is 500 atmospheres, which crushes a lot of oceanographic instruments.

For a perfect gas, the equation of state

$$p = \rho RT$$

combines with the hydrostatic MOM balance

$$p_z = -\rho g$$

to give

$$p_z = -gp / RT$$

so that the pressure depends on the integral of $T^{-1}$:

$$p = p_s \exp(-\frac{g}{R}\int_0^z \frac{dz'}{T(z')})$$

where $p_s$ is the surface pressure. For an isothermal atmosphere (T=const.=$T_0$), the pressure and density are simple exponential curves,

$$p = p_s \exp(-gz / RT_0)$$

decreasing vertically with an e-folding scale, the *scale height*, $H_s=RT_0/g$.

$H_s$ = 7.4 km at an average temperature typical of the troposphere, a Kelvin temperature 250K (=-23$^0$C), while $H_s \approx 8$ km at the ground, $T_0 \cong 280$K). For dry air, R=287.04 Joule kg$^{-1}$ $^0$C$^{-1}$ and for the water vapor component of moist air, R=461.50 Joule kg$^{-1}$ $^0$C$^{-1}$ (for different gases, R is equal to a universal constant, $R_* = 8314.36$ Joule kmol$^{-1}$ $^0$C$^{-1}$ divided by the molecular weight of the gas.

For the ocean, the typical difference in density from top to bottom is about 2%, so that the scale height, $H_s$, the distance over which the density would, say decrease to 1/e of its value at the bottom, is roughly 200 km…far greater than the depth of the ocean.

***Buoyancy frequency.*** The stratification into layers of differing density is one of the key properties of oceans and atmosphere. It makes the 'thin' fluids even thinner, in a dynamical sense, restricting movement to be principally horizontal. The measure of this stratification is the restoring force felt by a parcel displaced in the vertical, and then allowed to fall back to its 'rest' altitude.

The standard argument is this: the density of a fluid parcel displaced upward in a stratification will differ from its surroundings. The difference would be simply

$$\delta\rho = \rho_{parcel} - \rho_{surroundings} \approx \ \delta\rho \equiv \rho_{parcel} - \rho_{surroundings} \approx -\frac{d\rho_0}{dz}\delta z$$

if the fluid were incompressible. But in fact the fluid expands as it rises, its internal pressure being about the same as that of the surrounding fluid. If there is no heat exchange with its surroundings, the adiabatic or isentropic equation of state is appropriate

$$p = \exp(\eta / C_v)\rho^{\gamma}$$

where $\eta$ is entropy and $C_v$ is specific heat capacity at constant volume. The density difference is then

$$\begin{aligned}\delta\rho &= -\frac{d\rho_0}{dz}\delta z + \frac{\partial\rho}{\partial p}\Big|_{\eta}\frac{dp_0}{dz}\delta z \\ &= -\frac{d\rho_0}{dz}\delta z + c^{-2}\frac{dp_0}{dz}\delta z \\ &= -\frac{d\rho_0}{dz}\delta z - \rho_0 g c^{-2}\delta z\end{aligned}$$

where $c^2 \equiv \frac{\partial p}{\partial\rho}\Big|_{\eta}$ is the speed of sound in the fluid. For a perfect gas, $c^2 = \gamma RT$ where $\gamma = C_p/C_v$ is the ratio of the two specific heat capacities, $\gamma = 1.4$ for a diatomic gas like air; thus, c = 347 m $sec^{-1}$ at 300K.

Because the vertical density gradient is negative, the two terms compete, the compressibility reduces the restoring force.

The buoyancy frequency is simply the result of taking this restoring force and forming an equation for small up-and-down oscillations of the parcel.

$$\frac{d^2}{dt^2}\delta z + N^2\delta z = 0$$

$$N^2 = -\frac{g}{\rho_0}\left[\frac{d\rho_0}{dz} + \frac{\rho_0 g}{c^2}\right]$$

See the written discussion on the Boussinesq approximation for more. The ratio of the second term divided by the first term is

$$\frac{gH_s}{c^2} \approx 1$$

for air with a 10 km scale height, so compressibility cannot be neglected except for idealized, simplified models. For the oceans this ratio is between 1 and 2 as well.

***Potential temperature and potential density.*** The temperature, following a fluid parcel, changes due to adiabatic compression or expansion. If we imagine bringing fluid to a particular reference height, $z_{ref}$, the *potential temperature* θ is defined as the temperature that the parcel would have if moved adiabatically to $z_{ref}$. To relate θ and T, use the 1st Law of thermodynamics (the internal energy equation) written with entropy, η, as one of the two variables:

$$C_P \delta T = T\delta\eta + \frac{1}{\rho}\delta p$$

If δη=0 then

$$C_P \delta T / T = \frac{R}{\rho}\delta p$$

using p = ρRT. Integrate this equation from pressure p, temperature T to the reference pressure $p_r$ and the temperature at that level, which is θ:

$$\theta = T(\frac{p_r}{p})^{R/C_P} = T(\frac{p_r}{p})^{(\gamma-1)/\gamma}$$

where γ=1.4 for a perfect dry gas.

Now we can express the buoyancy frequency, N, in terms of θ. The parcel argument shows that, with the effects of adiabatic expansion already incorporated in θ,

$$\delta\rho = \rho\alpha(T_{parcel} - T_{surroundigs})$$

$$= \rho\alpha\frac{d\theta}{dz}\delta z$$

where $\alpha=-(1/\rho)\partial\rho/\partial T|_{p=const}$. For a perfect gas, α = 1/T. The buoyancy force on the parcel is -gραδT, and hence the buoyancy frequency, squared, is this

$$N^2 = (g/\rho)\delta\rho/\delta z$$

$$= g\alpha\frac{d\theta}{dz} \qquad (g/\rho)\delta\rho/\delta z$$

$$= \frac{g}{\theta}\frac{d\theta}{dz}$$

In ocean dynamics, potential temperature is defined based on the measured equation of state, which has nonlinear dependence on temperature, T.

*Potential density* is similarly defined as the density the fluid would have if removed to a particular reference pressure (‘altitude’), $p_r$ . Thus it is a function of position, time and reference pressure

$\rho_p(x,y,z,t,p_r)$

In oceanography, the choice of reference pressure is important, owing to the nonlinear dependence of density on (T,S,p). The parcel argument above carried out for potential density gives the simple result,

$$N^2 = -\frac{g}{\rho}\frac{d\rho_p}{dz}$$

## 2. Geostrophic balance

Unstratified fluid on a rotating planet is stiffened by the rotation, along lines parallel with the rotation vector, $\vec{\Omega}$. The Taylor-Proudman approximation for such a fluid is that

$$u_z = 0, v_z = 0 + O((RoH/L)^2) \qquad \vec{u} = (u,v,w) \tag{2.1}$$

if the rotation axis is aligned with the z-axis. The error estimate (O(…)) is the result of a scale analysis of the vorticity equation; when either Rossby number Ro or the aspect ratio H/L are small, the error tends to be very small (note, in the lab we often have H/L ~1 or greater and still have Taylor columns when Ro is small). On a rotating sphere, where the rotation axis is not parallel with the local vertical, very slow, unstratified, large-scale flows move in columns aligned with $\vec{\Omega}$, yet in many realistic flows it will become more accurate to retain only the local vertical component of rotation, neglecting the horizontal component.

Let us examine how the vertical velocity responds to this 'Taylor column' structure in u and v. For this uniform-density fluid, we differentiate one of the MASS conservation equations with respect to z:

$$\nabla \bullet \vec{u} = 0 \Rightarrow u_{xz} + v_{yz} = -w_{zz} = 0$$

after using (2.1). Thus the vertical velocity varies linearly with z (or is constant). Clearly, the stiffness of the fluid would favor keeping $w_z = 0$, and indeed the fluid will tend to follow paths of constant vertical thickness, for example flowing round a mountain rather than over it. But if it is forced to change its thickness, for example flowing over a long mountain ridge, it will do so with this linear dependence on z.

Geostrophic balance occurs for flows that vary gradually in space and slowly in time; That is, if the two Rossby numbers, Ro= U/fL, and 1/fT are both small. Then the horizontal momentum balance between the Coriolis force and pressure gradient force is quite accurate, and the pressure thus acts as a stream-function for the horizontal velocity. However accelerations still occur, and pressure variations along streamlines are often necessary to drive these: these pressure variations are small but dynamically essential. This is evident, for example in flow round a circular cylinder, which has high pressure at the fore- and aft stagnation points (where the velocity vanishes) and low pressure on the sides where the flow speeds up to round the cylinder. Bernoulli's equation holds for steady flows and describes these non-geostrophic effects.

Geostrophic balance is the simplified horizontal MOM equations

$$[2\vec{\Omega}\times\vec{u} = -\nabla p/\rho]_{horizontal}$$

In both atmospheric and oceanic dynamics the *dynamic height* is a useful variable. The geopotential function, $\Phi \cong gz$, where g is the combined acceleration due to gravity (inward, toward the Earth's center) and centrifugal (outward) forces. With hydrostatic balance,

$$\nabla p = -\rho\nabla\Phi \quad or$$
$$d\Phi = -dp/\rho$$

Let us relate the pressure gradient measured along level surfaces, z=const with the slope of the surfaces of constant pressure. For any function m(x,z),

$$\frac{\partial m}{\partial x}|_{p=const} = \frac{\partial m}{\partial z}\frac{dz}{dx}|_{p=const} + \frac{\partial m}{\partial x}|_{z=const}$$

Now if m is pressure itself,

$$\frac{\partial p}{\partial x}|_{z=const} = -\frac{\partial p}{\partial z}\frac{dz}{dx}|_{p=const}$$
$$= \rho g\frac{dz}{dx}|_{p=const}$$

We now can rewrite the geostrophic balance in terms of the geopotential as

$$\boxed{\begin{aligned}2\vec{\Omega}\times\vec{u} &= -g\nabla z\,|_{p=const}\\ &= -\nabla\Phi\,|_{p=const}\end{aligned}} \qquad (2.2)$$

Maps of the atmospheric circulation most commonly use the *geopotential height*, Z, in meters, which is simply

$$Z = \Phi/g_c = gz/g_c$$

where often $g_c$=9.8 m $sec^{-1}$ is chosen as a reference value of g (Gill, p.46, 213). Using either Φ or Z, the density does not appear in this version of geostrophic balance which is of advantage, because density varies greatly with altitude dependent. Maps of Φ or D give information about the velocity without knowing the density.

With respect to geostrophic balance in a stratified fluid Φ or Z or p' plays the role of the free-surface displacement, η, in a single layer of uniform density fluid.

***Stratified fluid, thermal wind.*** Fluid remains 'stiff' when it is stratified: it takes strong forces and great energy to stretch it vertically. Yet stratification allows the horizontal velocities to vary in z. The combination of hydrostatic- and geostrophic balance gives the thermal wind equation. The basic thermal wind result follows from combining the geostrophic and hydrostatic MOM equations, where we approximate ρ in the pressure gradient term by $\rho_0$, the density of the background mean state. This gives

$$-fv = -\frac{1}{\rho_0} p'_y \qquad (a)$$

$$fu = -\frac{1}{\rho_0} p'_x \qquad (b)$$

$$p'_z = -g\rho' \qquad (c)$$

$$\rho = \rho_0(z) + \rho'; \;\; p = p_0(z) + p'$$

So $\partial(a)/\partial z + (1/\rho_0)\,\partial(c)\partial y$ and $\partial(b)/\partial z - (1/\rho_0)\,\partial(c)/\partial x$ gives

$$fv_z = -\frac{g}{\rho_0}\rho'_x$$

$$fu_z = \frac{g}{\rho_0}\rho'_y$$

or

$$\vec{f} \times \vec{u}_z = \frac{g}{\rho_0}\nabla\rho' \qquad \vec{f} = 2\vec{\Omega}\,\sin(latitude)$$

which is the usual thermal-wind equation.

A more accurate version may be found by keeping the full density in all terms:
Differentiate (2.2) with respect to z:

$$\begin{aligned}
2\vec{\Omega} \times \frac{\partial \vec{u}}{\partial z} &= -\frac{\partial}{\partial z}\frac{1}{\rho}\nabla p \\
&\quad -\frac{1}{\rho}\nabla\frac{\partial p}{\partial z} + \frac{\nabla p}{\rho^2}\frac{\partial \rho}{\partial z} \\
&= \frac{g}{\rho}\nabla\rho + \frac{\nabla p}{\rho^2}\frac{\partial \rho}{\partial z} \\
&= \frac{g}{\rho}\nabla\rho - \frac{\nabla z\,|_p \,\frac{\partial p}{\partial z}}{\rho^2}\frac{\partial \rho}{\partial z} \\
&= \frac{g}{\rho}(\nabla\rho\,|_z + \nabla z\,|_p\,\frac{\partial \rho}{\partial z}) \\
&= \frac{g}{\rho}\nabla\rho\,|_{p=const}
\end{aligned}$$

In the last two lines we use the same argument made earlier, which is best clarified by sketching the sloping isobaric surface and horizontal, z=const. surfaces. This is the thermal wind equation (note that if the density is constant we recover the Taylor-Proudman approximation. Taking the density gradient along constant height (z) surface instead of constant pressure surface, gives the more familiar thermal wind equation.

$$\frac{g}{\rho}\nabla\rho\,|_{p=const} = \frac{g}{\rho}\nabla\rho\,|_{z=const} + O(H/H_S)$$

Surfaces of constant pressure are much more level than surfaces of constant density, for motions with vertical scale less than the density 'scale-height' $H_s = -\rho/(d\rho/dz) = g/N^2$:

$$\frac{dz/dx|_p}{dz/dx|_\rho} \sim H/H_S$$

This is worked out by writing the ratio as

$$\frac{p_x/p_z}{\rho_x/\rho_z}$$

and then substituting geostrophic, hydrostatic estimates of the various terms and using the above definition of the density scale height. Thermal wind balance is one keystone of dynamical and observational meterology and oceanography: it converts density information into velocity information. Very often it is easier to measure density than pressure or velocity. Various other forms of the thermal wind equation exist: Margules' relation (for a fluid made up of layers, each of uniform density); the dynamic 'thickness' equation which is a vertical integral of thermal wind; relations involving velocity veering and backing with height, on the one hand and horizontal density advection, on the other.

The physical nature of the buoyancy twisting term is this: imagine a small spherical region of the fluid. The pressure force pressing on this sphere from the fluid outside is always perpendicular to the surface of the sphere, and hence the force vectors all pass through the geometric center of the sphere. This is also the center of mass if the fluid has uniform density, and in this circumstance the pressure forces exert no torque about the center of mass. However, if the fluid is stratified, its center of mass will be *below* the geometric center, and then the pressure force vectors do exert a torque about the center of mass, which will drive angular momentum change and vorticity change (equal to the integral of $\vec{r}\times\vec{F}$, where $\vec{r}$ is the radius vector from the center of mass, and $\vec{F}$ is the pressure force).

***Veering and backing.***

***Margules' relation for a layered-density fluid.***

These two ideas follow directly from thermal-wind thinking: see Gill Ch. 7. If the horizontal velocity is pointed in the direction of a horizontal density gradient (or has a component in that direction), then thermal wind tells us that the velocity at height z + δz will be rotated left or right from $\vec{u}(z)$. In fact $\delta\vec{u}$ lies *along* the contours of constant density. This causes the velocity vector to rotate counterclockwise ('left' in northern hemisphere, or cyclonically) with increasing height if one faces the less dense fluid ('warm air'); this is known as 'backing'. If one is facing more dense fluid ('cold air') the horizontal velocity rotates clockwise ('right', anticyclonically) with increasing z, and this is 'veering'. With your back to the wind and to the cold air you have 'cold advection' and conversely with your back to the wind and warm air, 'warm advection', in the sense that the horizontal part of $\vec{u}\bullet\nabla\rho'$ is positive or negative. Draw a sketch to see this!

## 3. Vorticity dynamics

Vorticity dynamics provides a bridge between homogeneous (uniform density) and stratified fluids. The thermal wind equation is, in fact, a horizontal vorticity equation with several terms neglected. Geostrophic-adjustment calculations show that if one starts with a tilted stratification yet no velocity, the fluid will 'slump', the denser fluid flowing underneath the less dense fluid (and heavy fluid sinking while light fluid rises): this is a creation of horizontal vorticity. However after a fraction of a day, Coriolis forces take hold and rotating the horizontal velocity vector, bringing the fluid into a geostrophic balance. Transient inertial oscillations and internal waves may be superimposed on top of this balanced mean flow. It is these initial-value problems that provide a way of deciding the sign of the thermal wind flow.

The potential vorticity ('q' or 'PV') has been defined as an angular-momentum like quantity relating to the vertical vorticity. In one-layer, uniform-density fluid,

$$\frac{Dq}{Dt} = 0 + forcing - dissipation; \quad q = \frac{f+\zeta}{h}$$

where f is the vertical component of $2\vec{\Omega}$, $\zeta$ is the vertical vorticity and h the vertical layer thickness. If h has only small variations about its mean, then

$$q \cong \frac{f}{H} + \frac{\zeta}{H} - \frac{f\tilde{h}}{H^2} - \frac{f\eta}{H^2} \quad h = H + \tilde{h}(x,y) + \eta(x,y,t) \tag{2.3}$$

where $\tilde{h}$ is the contribution of bottom topography (positive for a valley, negative for a hill) and $\eta$ is the height of the fluid's free surface, relative to its mean. It is this restricted topographic amplitude that led to the p.d.e. for $\eta$, for geostrophic flows and transient adjustment.

We now want to create a potential vorticity equation for a stratified fluid, and express it in terms of a single dependent variable (as we had a p.d.e. for the single variable $\eta$, with a one-layer fluid). Stratified fluid can be approximated by a series of layers of uniform density. If the density difference from one layer to the next is the same for all layers, then their thicknesses, h, will differ. They differ such that a weak stratification has thick layers, a strong stratification thin layers. In fact we can write

$$h = \frac{\delta\rho}{\partial\rho / \partial z}$$

which a sketch of $\rho(z)$ should clarify. Therefore the 'topographic' terms contributing to q can be rewritten in terms of the density or the pressure. This approach is just what one does in building a numerical model, in this case suggesting an *isopycnal model* in which grid points ride with moving layers rather than being fixed points along the z-axis.

The result is a new expression for the potential vorticity:

$$q = \rho^{-1}(f+\zeta)\frac{\partial \rho}{\partial z} \qquad (2.4)$$

Now $\partial\rho/\partial z$ has a mean part, the background stratification, plus a perturbation part, $\partial\rho'/\partial z$, due to the moving fluid. The hydrostatic vertical MOM balance tells us that this is just

$$\frac{\partial \rho'}{\partial z} = -\frac{1}{g}\frac{\partial^2 p'}{\partial z^2}$$

With $\zeta$ also expressed in terms of p', we arrive at an expression for potential vorticity in terms of p' along: (2.4) becomes

$$q = \left(\frac{f}{\rho_0} + \frac{\nabla^2 p'}{\rho_0 f}\right)\frac{d\rho_0}{dz} - \frac{f}{\rho_0 g}\frac{\partial^2 p'}{\partial z^2}$$
$$= \frac{f}{\rho_0}\frac{d\rho_0}{dz} - \frac{N^2}{g}[\nabla^2 p' + \frac{f^2}{N^2}\frac{\partial^2 p'}{\partial z^2}]$$

where $N^2 = -g/\rho_0\, d\rho_0/dz$. Apart from the multiplier (a function of z only) and additive function of z, the active part of q is nearly a 3-dimensional Laplacian

$$[\nabla^2 p' + \frac{f^2}{N^2}\frac{\partial^2 p'}{\partial z^2}]$$

and if fact if N is independent of z, we can stretch the vertical coordinate by defining

$$\tilde{z} = zN/f$$

from which the potential vorticity (to within a constant) becomes

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2} \equiv \Delta p'$$

which is just the Laplacian operator in three dimensions. This remarkable relation suggests we think of the geostrophic pressure as being a field 'radiated' from 'charges' of potential vorticity. The ideas of spectral filtering apply: p' is a much smoother function of x, y, and z than is q. In fact because q is conserved, but for dissipation and external forces, it will tend to be pulled into long thin sheets like a color tracer in a fluid. This helps to explain the small spatial scale that q takes on, even when the velocity and pressure eddies have large scale.

***Synoptic-scale, quasi-geostophic potential vorticity equation.*** The title is unfortunately long. We use these ideas to write the full, time-dependent equation for p' or $\psi$, under the assumption that N varies only in z, and that the flow is nearly ('quasi-') geostrophic and hydrostatic. We can still have horizontal density gradients and thermal wind (vertical shear of the horizontal velocity) yet only so long as $\partial\rho'/\partial z \ll d\rho_0/dz$. The derivation parallels that above, yet begins from the vertical vorticity equation. Pedlosky (Sec 6.5) has a good treatment, with careful scaling.

We establish first that *buoyancy twisting generates horizontal vorticity, but not vertical vorticity* for quasi-geostrophic flow. The buoyancy term in the vorticity equation is

$$\nabla p \times \nabla \rho / \rho^2$$

Now under hydrostatic scaling [$(H/L)^2 << 1$] both $\nabla p$ and $\nabla \rho$ are almost vertical vectors. Thus their cross-product is nearly horizontal, and so only the horizontal vorticity equations are forced by horizontal density gradients. Establishing the accuracy of this statement is a straightforward exercise in scale analysis of the horizontal vorticity equation.

The vertical vorticity equation becomes, with this simplification,

$$\frac{D^{(g)}\zeta}{Dt} = (f+\zeta)\frac{\partial w}{\partial z}$$

where the $D^{(g)}/Dt$ operator involves only the *horizontal* advective terms:

$$\begin{aligned}\frac{D^{(g)}\zeta}{Dt} &\equiv \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} \\ &= \frac{\partial \zeta}{\partial t} - \frac{\partial p'}{\rho_0 f \partial y}\frac{\partial \zeta}{\partial x} + \frac{\partial p'}{\rho_0 f \partial x}\frac{\partial \zeta}{\partial y} \qquad (2.5) \\ &= \frac{1}{\rho_0 f}\left[\frac{\partial \nabla^2 p'}{\partial t} - \frac{\partial p'}{\rho_0 f \partial y}\frac{\partial \nabla^2 p'}{\partial x} + \frac{\partial p'}{\rho_0 f \partial x}\frac{\partial \nabla^2 p'}{\partial y}\right]\end{aligned}$$

We dropped the $w\partial\zeta/\partial z$ term on the grounds that $w/u \sim Ro\, H/L$.
This expression can be simplified a bit by introducing the geostrophic streamfunction,

$$\psi = p'/\rho_0 f$$

whereupon we have

$$\begin{aligned}\frac{D^{(g)}\zeta}{Dt} &= \frac{\partial \nabla^2 \psi}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} \\ &= \frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2 \psi)\end{aligned}$$

where J(a,b) is the Jacobian operator, sometimes written as $J(a,b) = \dfrac{\partial(a,b)}{\partial(x,y)} = \nabla a \times \nabla b \bullet \hat{z}$

Now, to proceed, if we express $\zeta$ and the D/Dt operator in terms of p' or $\psi$, we need to do the same for vertical velocity, w. The first simplification is to drop the right-hand $\zeta$ term on the grounds that $\zeta/f \sim U/fL = Ro$ which is small. The second is to assume the fluid to be incompressible. The MASS conservation equation is then

$$\frac{D\rho}{Dt} = 0 \qquad \rho = \rho_0(z) + \rho'$$

$$\frac{D^{(g)}\rho'}{Dt} = -w\frac{d\rho_0}{dz}$$

The density is conserved following a fluid parcel, yet following along a horizontal plane the density changes as fluid advects vertically through that plane. *The vertical velocity in a quasi-geostrophic flow is,* from the equation just above,

$$
\begin{aligned}
w &= -\left(\frac{1}{d\rho_0/dz}\frac{D^{(g)}\rho'}{Dt}\right) \\
&= \left(\frac{1}{g\,d\rho_0/dz}\frac{D^{(g)}\partial p'/\partial z}{Dt}\right) \\
&= -\frac{1}{\rho_0 N^2}\frac{D^{(g)}}{Dt}\frac{\partial p'}{\partial z}
\end{aligned}
\qquad (2.6)
$$

In terms of the geostrophic streamfunction, ψ, this is

$$w = -\frac{f}{\rho_0 N^2}\frac{D^{(g)}}{Dt}\frac{\partial(\rho_0\psi)}{\partial z}$$

The vertical stretching of vorticity becomes from (2.6)

$$f\frac{dw}{dz} = -f\frac{d}{dz}\left(\frac{1}{\rho_0 N^2}\frac{D^{(g)}(\partial p'/\partial z)}{Dt}\right)$$

Now what makes the derivation continue is that the D/Dt operator can be slipped out of the expression in brackets, because it has no z-derivative in it.

$$f\frac{dw}{dz} = -f\frac{D^{(g)}}{Dt}\rho_0\frac{d}{dz}\left(\frac{1}{\rho_0 N^2}\frac{\partial p'}{\partial z}\right)$$

We finally have

$$
\boxed{
\begin{aligned}
&\frac{D^{(g)}q^{(qg)}}{Dt} = 0 \\
&q^{(qg)} = (\rho_0 f)^{-1}\left[\nabla^2 p' + \rho_0\frac{\partial}{\partial z}\left(\frac{f^2}{\rho_0 N^2}\frac{\partial p'}{\partial z}\right)\right] \\
&\textit{in terms of streamfunction:} \\
&q^{(qg)} = \left[\nabla^2\psi + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0 f^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]
\end{aligned}
}
\qquad (2.7)
$$

and, as derived heuristically above, the potential vorticity is closely related to the Laplacian of the pressure field. The expression in brackets is the *quasi-geostrophic potential vorticity*, derived for synoptic scales, neglecting the Earth's curvature (the β-effect). [A note on units: our one-layer model had a potential vorticity q = (f + ζ)/h, with units 1/LT. The quasi-geostrophic potential vorticity, $q^{(qg)}$, above has units 1/T. This is inconvenient, and we could divide $q^{(qg)}$ by a reference depth to make them agree, but this is not usually done.]

The gradient of ψ has three components, which neatly describe both velocity and motion-induced density fields:

$$\begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix} = \begin{bmatrix} v \\ -u \\ -\dfrac{g}{f\rho_0}\rho' \end{bmatrix}$$

The vertical thickness contribution to PV has been neatly expressed as a z-derivative of p' or ψ. This allows us to quickly compare the two contributions to PV, from relative vorticity and thickness. Their ratio has a magnitude

$$\frac{f^2L^2}{N^2H^2} \equiv \frac{L^2}{\lambda^2}$$

which is the familiar ratio of horizontal length scale to Rossby deformation radius, squared. Recall that this parameter expresses the ratio APE/KE, available potential energy to kinetic energy.

***Boundary conditions.*** A mathematical problem is not complete until we specify boundary conditions (and if appropriate, initial conditions). If we completely neglect viscous effects, the 'inviscid' condition on a rigid boundary is

$$\vec{u} \bullet \hat{n} = 0$$

for a unit vector $\hat{n}$ normal to the boundary. For a flat, level, rigid boundary at z = -H this is

$$\frac{D^{(g)}}{Dt}\frac{\partial p'}{\partial z} = 0$$

in terms of the geostrophic pressure. It says that the density perturbation is conserved, following the purely horizontal fluid motion at that boundary. If such a level, horizontal boundary has uniform temperature at some time, it will remain so as the fluid moves (in absence of any forcing and diffusion terms). In this case the boundary condition is

$$\frac{\partial p'}{\partial z} = 0$$

on z = -H. For a steady flow, in the presence of sloping boundary topography, the above boundary condition becomes

$$-\vec{u} \bullet \nabla h = w$$

$$\frac{-1}{f\rho_0}\hat{z} \times \nabla p \bullet \nabla h = w$$

$$= \frac{-1}{\rho_0 N^2}\vec{u} \bullet \nabla \frac{\partial p'}{\partial z} \quad on\ z = -h(x,y)$$

$$\frac{-1}{f\rho_0}\hat{z} \times \nabla p \bullet \nabla\left(h - \frac{1}{\rho_0 N^2}\frac{\partial p'}{\partial z}\right) = 0$$

which is very complicated way of saying something simple: that the fluid next to the boundary keeps the same potential density, as it follows a streamline. $\hat{z}$ is a vertical unit vector. Another way of writing this is simply

$$\rho' = -\tilde{h}\frac{d\rho_0}{dz} \qquad \rho = \rho_0 + \rho'$$

where $\tilde{h}(x,y)$ is the topography. This uses the first Taylor series approximation to the statement $D\rho/Dt=0$, for small topographic height.

***Stratified, rotating flow over sinusoidal mountains.*** Suppose that there is a strong, steady zonal flow, U, uniform in space which encounters a chain of mountain ridges and valleys. Let us solve the PV equation to give the pressure, density and velocity of the flow. We have

$$\frac{D^{(g)}q}{Dt} = 0 \qquad q = [\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z})]$$

$$or \quad U\frac{\partial}{\partial x}[\nabla^2\psi + \frac{\partial}{\partial z}(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z})] \cong 0$$

The full, more exact equation would be $\vec{u}\bullet\nabla q = 0$ but here we keep only the dominant term $Uq_x = 0$. In the end this puts a restriction on the allowable mountain heights. Therefore the PV in brackets is conserved following along a streamline and here, approximately, along a latitude circle, y=const. If all of the fluid originates upstream over a flat bottom, q will be the same constant on all streamlines. Take this constant to be zero. The streamfunction representation of the velocity, to be complete, should be written

$$\begin{aligned}\psi &= \Psi(y) + \psi'(y,z)\\ &= -Uy + \psi'(y,z)\end{aligned}$$

Now the boundary condition is that with

$$h = H + \tilde{h}(x) = H - d\sin k_0\, x$$

the lower boundary lies at

$$z = -H + d \sin k_0 x.$$

Note the minus sign in the definition of h; here the ridge crests are where $d \sin k_0 x$ is positive, rather than the convention used earlier. Hence the boundary condition $\vec{u}\bullet\hat{n} = 0$ on the sloping boundary, with unit normal vector $\hat{n}$ becomes

$$w = -U\frac{d\tilde{h}}{dx}$$

$$U\frac{\partial}{\partial x}\frac{-f}{N^2}\frac{\partial\psi'}{\partial z} = -U\frac{\partial}{\partial x}\tilde{h}$$

or

$$\frac{f}{N^2}\frac{\partial\psi'}{\partial z} = +\tilde{h}$$

which we would apply at the height of the topography, z = -h(x,y). Instead we apply at the mean position of the lower boundary, z = H. This is a 'linearization of the boundary condition' which is familiar in fluid mechanics, and consistent with the other approximations made.

The topography is independent of the y-direction, so assume $\partial^2\psi/\partial y^2 = 0$. We have

$$q = \frac{\partial^2 \psi'}{\partial x^2} + \frac{f^2}{N^2}\frac{\partial^2 \psi'}{\partial z^2} = 0$$

with the boundary condition

$$\frac{f}{N^2}\frac{\partial \psi'}{\partial z} = \tilde{h} = -d \sin k_0 x \quad on \;\; z = -H$$ (… sign correction here)

The single sine-wave component in the boundary condition suggests a solution of the same form. Try $\psi' = M(z) \sin k_0 x$, whence

$$\frac{d^2 M}{dz^2} - k_0^2 \frac{N^2}{f^2} M = 0 \quad with \quad \frac{dM}{dz} = -N^2 d / f \quad on \;\; z = -H$$

The solution is

$$M = \frac{N^2 d}{\kappa f} e^{-\kappa(z+H)}$$ (…signs corrected here )

$$\kappa = k_0 N / f$$

Adding the zonal-mean flow (which is somewhat decoupled from the p' solution), the complete solution is

$$\psi = \Psi + \psi' = -Uy + \frac{Nd}{k_0} e^{-\kappa(z+H)} \sin k_0 x \quad \kappa = k_0 N / f$$ (…signs corrected here )

The flow-induced fields of pressure, density and velocity decrease exponentially with height above the mountains. The streamlines $\psi'$ = const. represent a meandering of the low-level flow, with high pressure and anticyclonic (- vorticity) over the ridges where h (the layer thickness) is minimum, cyclonic (+ vorticity) over the valleys where h is biggest. But the meandering of the zonal flow decreases with altitude, so that the effect of the mountains on the flow is limited to a layer of fluid of thickness fL/N = $f/Nk_0$, once again relating to the Rossby deformation radius. If the fluid has a finite depth, with a rigid boundary above, the flow will tend to be barotropic (independent of z) if L >> NH/f, just as in the one-layer theory without stratification; this is worked out below.

Close to the lower boundary, the streamlines or isobars, are given by setting $\psi'$=constant in the above solution:

$$y = \frac{Nd}{k_0 U} \sin k_0 x + const.$$

which is interesting, because the amplitude of the meandering streamline pattern is independent of f (even though Coriolis effects are essential in creating it). This is a good opportunity to test ones intuitive ideas about the flow, based on scale analysis. For example, the vertical vorticity, $\zeta$, is

$$\zeta = \nabla^2 \psi' = -k_0 N d \exp(-\kappa(z+H)) \sin k_0 x$$

The value of $\zeta$ at the bottom of the fluid, near the topography is independent of f. What result would we have expected without solving the mathematics? As we picture a fluid parcel flowing over the mountain valleys and ridges, the vertical thickness h' will have to change by a fraction $\delta h'/h' \sim d/H_1$ where $H_1$ is the height scale of the motion: $H_1$ is just $1/\kappa$. With $\kappa = k_0 N/f$, we have a fractional thickness change $\delta h'/h' = d\kappa = dNk_0/f$. PV conservation produces a vertical vorticity change $\zeta \sim f\, \delta h'/h' = dNk_0$ which agrees perfectly with the solution derived above.

Thus the stratification provides an effective upper boundary at a height fL/N above the lower boundary, so that the vortex stretching is greater (and the meandering of the flow is greater) with stratification than in a one-layer fluid of constant density and the same depth. Both the meandering pattern and the vorticity are independent of f (at the lowest levels), because when f is increased it would tend to increase vortex stretching but at the same time the vertical scale over which stretching occurs will increase, just canceling the direct effect of increasing f.

The density field is in phase with the topography, being disturbed in a sinusoidal pattern with cold or dense water directly over the ridges, and this relation unchanging with increasing z. The pressure field has high pressure over the valleys and low pressure over the ridges which is consistent with the pressure calculated from the Bernoulli equation: this can be viewed as a correction to the geostrophic pressure.

These phase relations are very important, and as you add things to this model, very soon you will see the pressure pattern shift downstream, so that low pressure occurs over the topographic downslope and high pressure occurs over the upslope. This is 'lee cyclogenesis' and it is a prominent feature of the large-scale atmospheric circulation, the ocean circulation in the Southern Ocean, and in small scale internal waves generated by flow over (small-scale) mountains.

The direction of the flow changes with z, as it is undisturbed and zonal far above the topography. This direction change has several implications.

***Upper boundary effect.*** We have neglected to this point the effect of the upper boundary, supposing it to be 'far away', and choosing the exp(κz) solution that dies away with increasing height rather than the exp(-κz) solution that would grow exponentially large. Now consider the effect of the upper boundary, where the boundary condition,

w = 0 on z = 0

gives

$$\frac{\partial \psi'}{\partial z} = 0 \qquad \text{on } z = 0.$$

This is said most simply as ρ' = 0 on z = 0, which follows from the connection between ρ' and the vertical advection of the basic density gradient (2.6). Now we modify the solution for the vertical eigenfunction, M(z). To satisfy this new boundary condition we need the exp(-κz) solution. We can assume

$$\psi' = A\exp(\kappa z)\sin k_0 x + B\exp(-\kappa z)\sin k_0 x$$

or, what is the same, use the combinations of real exponential functions known as hyperbolic-trigonometric functions,

$$\cosh(z) \equiv \tfrac{1}{2}(e^z + e^{-z}),\ \ \sinh(z) = \tfrac{1}{2}(e^z - e^{-z})$$

The one which has zero derivative at z = 0 is cosh(z). Thus if we write

$$M = A\cosh(\kappa z)$$

it automatically satisfies ∂ψ'∂z = 0 at the top. To satisfy the lower boundary condition, use the relation

$$\frac{d}{dz}(\cosh z) = \sinh z;$$

and find

$$\frac{dM}{dz} = \frac{-N^2 d}{f} \qquad at \ \ z = -H$$

$$-\kappa A\sinh(\kappa H) = \frac{-N^2 d}{f}$$

$$A = \frac{N^2 d}{f\kappa\sinh(\kappa H)} = \frac{Nd}{k_0\sinh(\kappa H)}$$

The complete solution, now satisfying both upper and lower boundary conditions is

$$\psi = -Uy + \frac{Nd}{k_0}\frac{\cosh(\kappa z)}{\sinh(-\kappa H)}\sin k_0 x \quad \kappa = k_0 N / f$$

which can be written in terms of exponential functions if one wishes to. If the fluid layer thickness is much greater than the vertical scale of penetration of the topographic 'wave', $\kappa^{-1}$, then cosh(κH) => ½ exp(κH), sinh(κH) => ½ exp(κH) and so

$$\psi => -Uy + \frac{Nd}{k_0}\frac{\exp(\kappa z)}{\exp(-\kappa H)}\sin k_0 x \quad \kappa = k_0 N / f$$
$$= -Uy + \frac{Nd}{k_0}\exp(\kappa(z+H))\sin k_0 x$$

which agrees with our solution above. This limit κH >>1 refers to short-wavelength topography, large N, small f, large H or some combination of these. Now, in the other extreme, κH << 1 the meandering pattern reaches to the top of the fluid. In this limit

$\cosh(\kappa H) => 1 + (\kappa H)^2$ , $\sinh(\kappa H) => \kappa H$ as $\kappa H => 0$,

and the solution becomes

$$\psi => \frac{Nd}{k_0}\frac{1+(\kappa z)^2}{(-\kappa H)}\sin k_0 x - Uy \quad \kappa = k_0 N / f$$
$$\cong \frac{fd}{k_0^2 H}\sin k_0 x \; - Uy$$

which is the solution for barotropic (depth-independent) flow of an unstratified (ρ = const) fluid over sinusoidal ridges. We have come full circle back to the one-layer flow when the mountain's wavelength is greater than the Rossby deformation radius NH/f is much greater than the fluid depth. This is a property of the 'f-plane', that is a fluid in which we take the Coriolis frequency to be constant. On the 'β-plane', or with a large-scale slope to the topography, Rossby waves will change these results very significantly.