

## Combined flow (wave-current) boundary layers

Pure wave boundary layers, as mentioned in the last lecture, are usually much smaller than current boundary layers because of their relatively short life span.

Before the 1980's, most scientists simply added the shear associated with waves and currents. However, Smith (1977) and Grant and Madsen (1979) formally extended the results discussed in the last lecture – illustrating the nonlinearity present in the interaction of waves and currents.

Smith's original analysis assumed that the waves and currents were oriented in the same direction. Because of the nature of this generally poor assumption, Grant and Madsen (1979) postulated the more general formulation –

$$\tau_m = |\tau_{wm} + \tau_c| \quad (1)$$

Where  $\tau_c$  is the current shear stress,  $\tau_{wm}$  is the maximum wave shear stress and  $\tau_m$  is the maximum total shear stress.

Considering the angle between the current and waves  $\phi_{wc}$ ,

$$\tau_m = \sqrt{(\tau_{wm} + \tau_c |\cos \phi_{wc}|)^2 + (\tau_c \sin \phi_{wc})^2} \quad (2)$$

which can be simplified to

$$\tau_m = \tau_{wm} \sqrt{1 + 2\mu |\cos \phi_{wc}| + \mu^2} \quad (3)$$

where  $\mu = \tau_c / \tau_{wm}$ . Considering  $\mu$  is usually small,  $\tau_m \approx \tau_{wm}$ .

The Grant-Madsen formulation again assumes a time-invariant eddy viscosity, this time, piecewise linear in the vertical

$$v_t = \begin{cases} \kappa u_{*m} z & z < \delta_{wc} \\ \kappa u_{*c} z & z > \delta_{wc} \end{cases} \quad (4)$$

where  $u_{*m} = \sqrt{\tau_m / \rho}$  and  $u_{*c} = \sqrt{\tau_c / \rho}$ .

One of the clever realizations of Grant-Madsen is that unsteadiness only exists in the direction of the waves. Assuming  $x$  is aligned with the propagation of waves and conserving momentum in the  $x$ -direction,

$$\rho \frac{\partial u_w}{\partial t} = \frac{\partial p_\delta}{\partial x} + \frac{\partial \tau_w}{\partial z} \quad (5)$$

where  $\tau_w$  is the shear stress associated with the waves only

$$\tau_w = \rho v_t \frac{\partial u_w}{\partial z} \quad (6)$$

where  $v_t$  is the eddy viscosity defined in (4). The solution of (5) is identical to Equation (11) in the last lecture –

$$\frac{\partial(u_w - u_b)}{\partial t} = \frac{\partial}{\partial z} \left[ \kappa u_{*m} z \frac{\partial(u_w - u_b)}{\partial z} \right] \quad (7)$$

The current velocity profile outside of the wave boundary layer responds only to the shear induced by the current, while inside it is a function of the total stress.

$$\frac{\tau_c}{\rho} = u_{*c}^2 = \begin{cases} \kappa u_{*m} z \frac{\partial u_c}{\partial z} & \text{for } z < \delta_{cw} \\ \kappa u_{*c} z \frac{\partial u_c}{\partial z} & \text{for } z > \delta_{cw} \end{cases} \quad (8)$$

The unidirectional component inside the wave boundary layer (i.e.,  $z < \delta_{cw}$ ) becomes

$$u_c = \frac{u_{*c}}{\kappa} \frac{u_{*c}}{u_{*m}} \ln \left( \frac{z}{z_0} \right) \quad (9)$$

outside the boundary layer, we assume that the

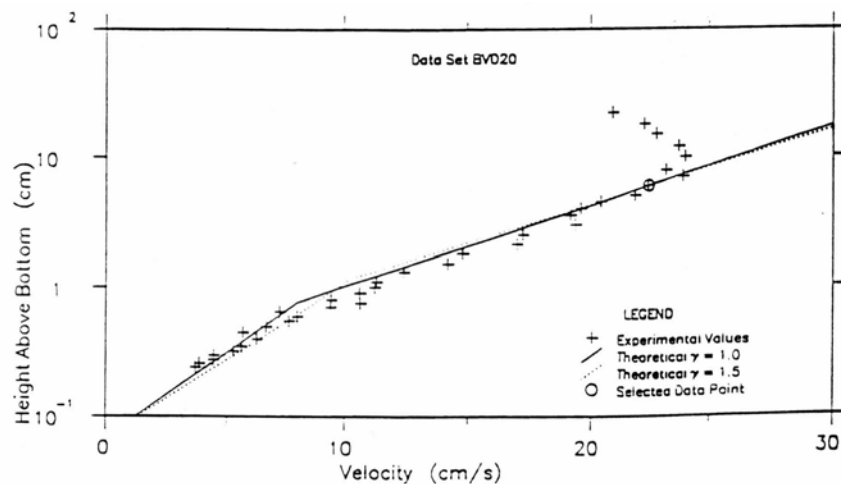
$$u_c = \frac{u_{*c}}{\kappa} \ln \left( \frac{z}{z_{0a}} \right) \quad (10)$$

where  $z_{0a}$  is sometimes called the **apparent bottom roughness**. It is really just a constant of integration.

You can solve for  $z_{0a}$  by matching the profile at the top of the boundary layer. Doing so, we find that (10) becomes

$$u_c = \frac{u_{*c}}{\kappa} \left[ \ln\left(\frac{z}{\delta_{cw}}\right) + \frac{u_{*c}}{u_{*m}} \ln\left(\frac{\delta_{cw}}{z_0}\right) \right] \quad (11)$$

As you might imagine, the discontinuity in the evaluation of the eddy viscosity leads to a ‘kink’ in the velocity distribution. Madsen notes that to smooth the profile requires significant effort without substantially improving the fit to laboratory data (shown below).



Plot taken from Madsen (1997) course notes. Experimental data obtained from Bakker and van Doorn (1978)

## *Calculation of combined shear stress*

To solve for the total combined shear stress  $\tau_m$ , we need to know the shear stress accompanying each component.

$$\mu = \frac{\tau_c}{\tau_{wm}} = \frac{u_{*c}^2}{u_{*wm}^2} \quad (12)$$

However, there will be an effect associated with the angle between the waves and the current. It is incorporated by

$$C_\mu = \sqrt{1 + 2\mu \cos \phi_{wc} + \mu^2} \quad (13)$$

To calculate the wave friction factor, in terms of both waves and currents, you must iteratively solve

$$\frac{1}{4\sqrt{f_{cw}/C_\mu}} + \log_{10} \left[ \frac{1}{4\sqrt{f_{cw}/C_\mu}} \right] = \log_{10} \left[ \frac{C_\mu A_{bm}}{k_r} \right] - 0.17 + 0.24 \left( 4\sqrt{f_{cw}/C_\mu} \right) \quad (14a)$$

for fully rough conditions (i.e.,  $k_r > 3.3\nu/u_{*m}$ )

$$\frac{1}{4\sqrt{4f_{cw}/C_\mu}} + \log_{10}\left[\frac{1}{4\sqrt{4f_{cw}/C_\mu}}\right] = \log_{10}\sqrt{\frac{C_\mu Re}{50}} - 0.17 + 0.06\left(4\sqrt{4f_{cw}/C_\mu}\right) \quad (14b)$$

for smooth beds (i.e.,  $k_r < 3.3\nu/u_{*m}$ ). Remember,  $Re = u_{bm}A_{bm}/\nu$ .

These equations are analogous to Equation (17) of the last lecture.

Once  $f_{cw}$  is solved for, we can then calculate the

$$\tau_{wm}/\rho = u_{*wm}^2 = f_{cw}u_{bm}^2/2 \quad (15)$$

and the combined total shear stress

$$\tau_m/\rho = u_{*m}^2 = C_\mu u_{*wm}^2 \quad (16)$$

The wave boundary layer is then modified accordingly.

$$\delta_{cw} = \kappa u_{*m}/\omega \quad (17)$$

You can use the wave boundary layer thickness to solve for the apparent roughness

$$z_{0a} = \delta_{cw} \left( \frac{z_0}{\delta_{cw}} \right)^{u_{*c}/u_{*m}} \quad (18)$$

If, for instance, you know the velocity at one location in the vertical (outside the wave boundary layer), you can solve (11) by

$$\kappa u_c(z_r) = \ln \left( \frac{z_r}{\delta_{cw}} \right) u_{*c} + \frac{u_{*c}^2 \ln(\delta_{cw}/z_0)}{u_{*m}} \quad (19)$$

Where the solution for  $u_{*c}$  is

$$u_{*c} = u_{*m} \frac{\ln(z_r/\delta_{cw})}{\ln(\delta_{cw}/z_0)} \left[ -\frac{1}{2} + \sqrt{\frac{1}{4} + \kappa \frac{u_c(z_r)}{u_{*m}} \frac{\ln(\delta_{cw}/z_0)}{[\ln(z_r/\delta_{cw})]^2}} \right] \quad (20)$$

which was found by solving (19) using the quadratic formula.

### ***Sample problem***

**Find:** The current shear velocity and bottom shear stress,  $u_{*c}$  and  $\tau_c$ , the maximum wave shear velocity and bottom shear stress,  $u_{*wm}$  and  $\tau_{wm}$ , as well as the maximum combined shear velocity and bed shear stress,  $u_{*wm}$  and  $\tau_m$ , for a combined wave-current flow over a flat bed.

Given: The wave is specified by its near-bottom maximum orbital velocity  $u_{bm} = 0.35$  m/s and period  $T = 8$  s. The current is specified by its magnitude  $u_c(z_r) = 0.35$  m/s at and direction  $\phi_{wc} = 45^\circ$  relative to the direction of wave propagation. The bottom is flat and consists of uniform sediment of diameter  $d = 0.02$  mm. You are at sea (i.e.,  $\rho = 1025$  kg/m<sup>3</sup>,  $\nu = 1 \times 10^{-6}$  m<sup>2</sup>/s).

Solution:

As outlined in the last lecture, we solve (iteratively) for the wave friction factor  $f_w$ , using  $\omega = 2\pi/T = 0.785$  1/s,

$A_{bm} = u_{bm}/\omega = 0.446$  m, and  $k_r = d = 2 \times 10^{-4}$  m, which were calculated from the given information. You always want to begin iteration by assuming the waves are dominant (i.e.,  $\mu = 0$ ;  $C_\mu = 1$  - this is most often the case).

After performing the iteration, we find

$$f_{cw} = f_w = 7.86 \times 10^{-3}$$

which yields the maximum shear velocity

$$u_{*m} = u_{*wm} \sqrt{C_\mu} = u_{*wm} = u_{bm} \sqrt{f_{cw}/2} = 2.19 \text{ cm/s}$$

which is identical to  $u_{*wm}$  because  $C_\mu$  is assumed equal to one. Our initial assumption of rough turbulent flow was correct.

Calculating the depth of the wave boundary layer from Equation (17), we find

$$\delta_{cw} = 1.12 \times 10^{-2} \text{ m (1.12 cm)}$$

With these values, and  $z_0 = k_r/30 = 6.67 \times 10^{-6} \text{ m}$ , Equation (20) is solved to give an estimate of the current shear velocity.

$$u_{*c} = 1.47 \times 10^{-2} \text{ m/s} = 1.47 \text{ cm/s}$$

which yields an estimate of the relative shear stress

$$\mu = 0.45$$

and a value of  $C_\mu = 1.36$

Iteratively solving for  $f_{cw}$ , now using the slightly altered Equation (14), but with the procedure outlined in the last lectures' notes, we find a new value of the combined wave-current friction factor

$$f_{cw} = 9.94 \times 10^{-3}$$

which yields

$$u_{*wm} = u_{bm} \sqrt{f_{cw}/2} = 2.47 \times 10^{-2} \text{ m/s} = 2.47 \text{ cm/s}$$

and

$$\tau_{wm} = \rho u_{*wm}^2 = 0.624 \text{ Pa}$$

We can also calculate the combined shear velocity and shear stress

$$u_{*m} = u_{*wm} \sqrt{C_\mu} = 2.88 \text{ cm/s and } \tau_m = \rho u_{*m}^2 = 0.850 \text{ Pa}$$

The total shear stress allows us to calculate the wave boundary layer thickness

$$\delta_{cw} = \kappa u_{*m} / \omega = 1.47 \text{ cm}$$

Finally, we calculate the shear velocity and shear stress associated with the current, using Equation (20) and the definition of shear velocity to obtain

$$u_{*c} = 1.63 \text{ cm/s}$$

which yields  $\mu = (u_{*c} / u_{*wm})^2 = 0.44$ , which is sufficiently close to our initial guess of 0.45, to use the solutions given above.

There are other possible scenarios where you will close the iteration loop slightly differently, but the one described above, will probably be most common.

## *The stratifying influence of sediment concentration*

Clearly, near the bed, the sediment concentration in most natural systems is enough to affect turbulence properties.

Glenn and Grant (1987) have formulated a model based upon the idea that entrainment of sediment alters the density of the water column such that it affects the vertical component of the eddy viscosity.

That is, they adapt the eddy viscosity model using a ‘stability factor’ based upon the sediment concentration in the water column.

The approach is elegant, however, in most situations the addition of sediment in the water column makes the turbulence strongly inhomogeneous (invalidating the eddy viscosity model). In many cases, it can actually make fluid non-Newtonian. If either of these effects is dominant (and they usually are for silts and clays), the approach of Glenn and Grant (1987) is no better than original Grant-Madsen.

## *Second-generation boundary-layer models*

As discussed above, the primary limitation of the Grant-Madsen approach is the simplification of the turbulence with an eddy viscosity.

It is beyond the scope of this course to discuss higher-order turbulence closure schemes ( $k$ - $\varepsilon$ , LES and DNS); however, it is

important to note that these models are on the horizon and that they will most likely be superior to Grant-Madsen. These models will eventually have to deal with the complexities discussed by Glenn and Grant (1987), if they are to attempt to model natural systems.