In the longshore lecture notes, we begin with

$$S_{xy} = En \sin\alpha \cos\alpha \qquad (1)$$

We can rewrite (1) as

$$S_{xy} = \left(EC_g \cos\alpha\right)\left(\sin\alpha / C\right) \qquad (2)$$

Where the first term on the RHS is the wave energy flux per unit shoreline length, and the second term is a constant according to Snell's Law of refraction.

We want to know how the first term varies as we approach the shoreline – as it produces the 'thrust' of the longshore current. To do this, we take the derivative of (2)

$$\frac{\partial S_{xy}}{\partial x} = \frac{\sin\alpha}{C} \frac{\partial\left(EC_g\right)\cos\alpha}{\partial x} \qquad (3)$$

Remember, $\sin\alpha / C$ is a constant.

From linear wave theory, we know

$$E = \frac{\rho g H^2}{8} \qquad (4)$$

This was incorrect in the original version of the notes (and denoted there as Equation 6).

We now assume shallow water waves (i.e., $C_g = \sqrt{gh'}$, $n = 1$). Knowing that the wave height H is constrained by the local water depth $h' = \bar{\eta} + h$, by the formulation –

$$\gamma = \left(\frac{H_{rms}}{h'}\right)_{at\ breaking} = 0.42 \tag{5}$$

The wave field ultimately becomes 'saturated' such that (5) is true for all of the waves.

Substituting (5) into (4) and (4) into (3), we find

$$\frac{\partial S_{xy}}{\partial x} = \frac{\rho g^{3/2}\gamma^2 \sin\alpha\cos\alpha}{8C}\frac{\partial}{\partial x}\left[\left(\bar{\eta}+h\right)^{5/2}\right] \tag{6}$$

Remember,

$$\frac{\partial\bar{\eta}}{\partial x} = -\left[\frac{1}{1+8/3\gamma^2}\right]\frac{\partial h}{\partial x} \tag{7}$$

So the derivative in (6) becomes

$$\frac{\partial}{\partial x}\left[\left(\bar{\eta}+h\right)^{5/2}\right] = \frac{5}{2}\left(\bar{\eta}+h\right)^{3/2}\frac{\partial\left(\bar{\eta}+h\right)}{\partial x} = \frac{5}{2}\left(\bar{\eta}+h\right)^{3/2}\left[1-\frac{1}{1+8/3\gamma^2}\right]\frac{dh}{dx}$$

Simplifying further –

$$\frac{\partial}{\partial x}\left[\left(\bar{\eta}+h\right)^{5/2}\right] == \frac{5}{2}\left(\bar{\eta}+h\right)^{3/2}\left[\frac{1}{1+3\gamma^2/8}\right]\frac{dh}{dx}$$

Substituting $\varsigma = 1/\left[1+\left(3\gamma^2/8\right)\right]$ into the above and knowing $C=\sqrt{g\left(\bar{\eta}+h\right)}$, we find

$$\frac{\partial S_{xy}}{\partial x} = -\frac{5\gamma^2}{16}\varsigma\rho g\left(\bar{\eta}+h\right)\frac{dh}{dx}\sin\alpha\cos\alpha \qquad \text{(8a)}$$

or

$$\frac{\partial S_{xy}}{\partial x} = \frac{5\gamma^2}{16}\varsigma\rho g h'\tan\beta\sin\alpha\cos\alpha \qquad \text{(8b)}$$

The result in Equation (8) makes the assumption of linear wave theory in a region dominated by breaking. This is a poor assumption. However, the result is useful in determining the functional relationship between the longshore current and the parameters that regulate it.