Radiation stress theory

This theory, originally posed in a series of papers by Longuet-Higgins and Stewart (1962, 1963, 1964) utilizes an analogy to EM waves and the pressure, or stress, they induce on surfaces.

Radiation stress, as we will define it, is directly analogous to Reynolds stresses in a unidirectional turbulent flow.

We begin with an Airy wave propagating in the direction x, where

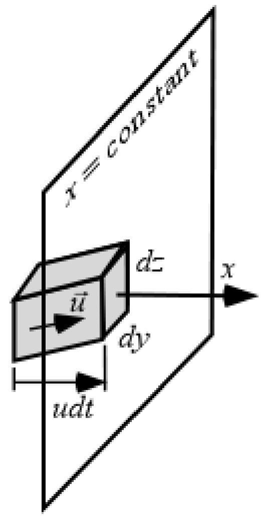
$$u = \frac{H\omega}{2\sinh kh}\cosh k(z+h)\cos(kx-\omega t) \tag{1a}$$

$$w = \frac{H\omega}{2\sinh kh}\sinh k(z+h)\sin(kx-\omega t) \tag{1b}$$

We are going to consider the flux of horizontal momentum across unit area of a vertical plane. I.e.,

$$M(x,t) = \int_{-h}^{\eta}\left(p + \rho u^2\right)dz \tag{2}$$

We will define the radiation stress S_{xx} as the mean value of $M(x,t)$ with respect to time, minus mean flux in the absence of waves. That is,



$$S_{xx} = \overline{\int_{-h}^{\eta} \left(p + \rho u^2\right) dz} - \int_{-h}^{0} p_0 dz \tag{3}$$

Essentially, S_{xx} consists of three terms

$$S_{xx}{}^{(1)} = \overline{\int_{-h}^{\eta} \rho u^2 dz} \tag{4a}$$

$$S_{xx}{}^{(2)} = \overline{\int_{-h}^{0} \left(p - p_0\right) dz} \tag{4b}$$

$$S_{xx}{}^{(3)} = \overline{\int_{0}^{\eta} p dz} \tag{4c}$$

where $S_{xx} = S_{xx}{}^{(1)} + S_{xx}{}^{(2)} + S_{xx}{}^{(3)}$.

Because Equation (1) was derived using a small-amplitude approximation, we can say that

$$S_{xx}^{(1)} = \overline{\int_{-h}^{0} \rho u^2 dz} = \int_{-h}^{0} \rho \overline{u^2} dz \qquad (5)$$

$S_{xx}^{(1)}$ is identical to the vertically integrated Reynolds stress in the streamwise direction (x).

We assume that there are no ‘non-hydrostatic’ effects. That is,

$$S_{xx}^{(2)} = \int_{-h}^{0} \left(\overline{p} - p_0\right) dz \qquad (6)$$

From incompressibility and continuity, we know that

$$\overline{p + \rho w^2} = -\rho g z = p_0 \qquad (7)$$

so

$$\overline{p} - p_0 = -\rho \overline{w^2} \qquad (8)$$

which means that $\overline{p}$ is generally less than zero.

It also means that $S_{xx}^{(2)} \leq 0$. Combining Equations (5), (6) and (8),

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \int_{-h}^{0} \rho \left(\overline{u^2 - w^2}\right) dz \qquad (9)$$

We can see that for anything other than deep-water waves, $S_{xx}^{(1)} + S_{xx}^{(2)} > 0$.

After integration of Equation (9), using the velocities defined in Equation (1), we find

$$S_{xx}^{(1)} + S_{xx}^{(2)} = \frac{\rho g H^2 kh}{4\sin 2kh} \qquad (10)$$

You should note that in deep water, that the addition of the first two terms of the radiation stress is zero (i.e., $\overline{u^2} = \overline{w^2}$). In shallow water, u dominates (i.e., $\overline{w^2} = 0$).

The sum of these first two terms is twice the kinetic energy.

The final term $S_{xx}^{(3)} = \int_0^{\eta} p\,dz$ can be simplified, if we assume that the pressure is hydrostatic –

$$p = \rho g(\eta - z) \qquad (11)$$

Substituting (11) into the integral for $S_{xx}^{(3)}$, we find

$$S_{xx}^{(3)} = \rho g \overline{\eta^2}/2 = \rho g H^2/16 \qquad (12)$$

which is simply the potential energy density.

Longuet-Higgins and Stewart (1964) cast the final form in terms of the total energy density E

$$S_{xx} = E\left(\frac{2kh}{\sinh 2kh} + \frac{1}{2}\right) \tag{13}$$

which is equivalent to Komar's Equation (6.13)

$$S_{xx} = \frac{1}{8}\rho g H^2\left[\frac{2kh}{\sinh 2kh} + \frac{1}{2}\right] \tag{14}$$

Radiation stress is actually a tensor. The transverse radiation stress S_{yy} has the same definition as S_{xx}, but for transverse direction (y)

$$S_{yy} = \overline{\int_{-h}^{\eta}\left(p + \rho v^2\right)dz} - \int_{-h}^{0} p_0 dz \tag{15}$$

We break up (15) in a similar way as before

$$S_{yy}^{(1)} = \overline{\int_{-h}^{\eta}\rho v^2 dz} = 0 \tag{16}$$

and

$$S_{yy}^{(2)} = \overline{\int_{-h}^{\eta} \rho w^2 dz} = S_{xx}^{(2)} \tag{17}$$

Likewise,

$$S_{yy}^{(3)} = \rho g \overline{\eta^2}/2 = S_{xx}^{(3)} \tag{18}$$

which results in

$$S_{yy} = \frac{\rho g H^2 kh}{8 \sinh 2kh} \tag{19}$$

It turns out that there are potentially non-zero fluxes due to flow of x-momentum across $y = \text{constant}$ planes (and vice versa). However, in our simple 2-D case with x aligned with direction of wave propagation, the cross-components of the radiation stress only depend on the quantity

$$S_{xy} = \overline{\int_{-h}^{\eta} \rho uv dz} \tag{20}$$

which is zero (v is zero). There is no dependence on the mean pressure in the cross-components (pressure is

In summary, we can express the radiation stress tensor **S**, for two-dimensional Airy waves with

$$\mathbf{S} = E\begin{pmatrix} \dfrac{2kh}{\sinh 2kh} + \dfrac{1}{2} & 0 \\ 0 & \dfrac{kh}{\sinh 2kh} \end{pmatrix} \tag{21}$$

where

$$E = \frac{\rho g H^2}{8}$$