Beginning with the equations of motion

$$\frac{\partial(u_w - u_b)}{\partial t} = \frac{\partial(\tau_w/\rho)}{\partial z} \qquad (1)$$

We begin with the assumption of a **time-invariant eddy viscosity**. Namely,

$$\tau_w = \rho \kappa u_{*wm} z \frac{\partial u_w}{\partial z} \qquad (2)$$

where u_{*wm} is the maximum shear velocity (i.e., the shear velocity associated with the maximum shear stress) and u_w is the velocity in the *x*-direction.

This assumption leaves the equations of motion to be

$$\frac{\partial(u_w - u_b)}{\partial t} = \frac{\partial}{\partial z}\left[\kappa u_{*wm} z \frac{\partial(u_w - u_b)}{\partial z}\right] \qquad (3)$$

Remember, u_b is NOT a function of z.

Grant and Madsen (1979) used Kelvin functions to solve (3). The solution to the Kelvin functions in the limit of $z_0\omega/\kappa u_{*wm} \rightarrow 0$ is:

$$u_w = \frac{u_{bm}}{\sqrt{\left[\ln\left(\frac{\kappa u_{*wm}}{\omega z_0}\right) - 1.15\right]^2 + \left(\frac{\pi}{2}\right)^2}} \ln\left(\frac{z}{z_0}\right) \cos(\omega t + \varphi) \qquad (4)$$

which can be simplified to

$$u_w = \frac{2u_{bm}}{\pi} \ln(z/z_0) \sin\varphi \cos(\omega t + \varphi) \qquad (5)$$

where u_{bm} is the maximum orbital velocity predicted from Airy theory, z_0 is related to the effective roughness of the bottom. The 'phase lead' φ is

$$\tan\varphi = \frac{\pi/2}{\ln\left(\frac{\kappa u_{*wm}}{\omega z_0}\right) - 1.15} \qquad (6)$$

Now we have the information required to solve Equation (2) for the wave shear stress.

$$\tau_w = \lim_{z\to 0}\left(\rho\kappa u_{*wm} z \frac{\partial u_w}{\partial z}\right) \qquad (7)$$

Evaluating the derivative of (4) and taking the limit of the remaining function, we find

$$\tau_w = \frac{\rho \kappa u_{*wm} u_{bm}}{\sqrt{\left[\ln\left(\frac{\kappa u_{*wm}}{z_0 \omega}\right)\right]^2 + \left(\frac{\pi}{2}\right)^2}} \cos(\omega t + \varphi) \tag{8}$$

or

$$\tau_w = \tau_{wm} \cos(\omega t + \varphi) \tag{9a}$$

where

$$\tau_{wm} = \frac{\rho \kappa u_{*wm} u_{bm}}{\sqrt{\left[\ln\left(\frac{\kappa u_{*wm}}{z_0 \omega}\right)\right]^2 + \left(\frac{\pi}{2}\right)^2}} \tag{9b}$$

Equation (9) is sufficient for solving the shear stress due to oscillatory forcing. However, when we will solve for combined boundary layers, we will need to solve for the wave friction factor f_w, so let's continue.

From the definition of the wave friction factor, we know

$$\tau_{wm} = f_w \rho u_{bm}^2 / 2 \tag{10}$$

where f_w is the wave friction factor u_{bm} is the maximum wave orbital velocity evaluated at the bed.

Substituting Equation (9b) into (10) and noting from the definition of shear velocity that

$$u_{*wm} = u_{bm}\sqrt{f_w/2} \tag{11}$$

we find

$$\kappa\sqrt{\frac{2}{f_w}} = \sqrt{\left[\ln\left(\frac{\kappa u_{bm}\sqrt{f_w/2}}{z_0\omega}\right)-1.15\right]^2 + \left(\frac{\pi}{2}\right)^2} \tag{12}$$

Grant and Madsen (1986) suggest the approximation to Equation (12). That formulation for rough flow is:

$$\frac{1}{4\sqrt{f_w}} + \log_{10}\left(\frac{1}{4\sqrt{f_w}}\right) = \log_{10}\left(\frac{A_{bm}}{k_r}\right) - 0.17 + 0.96\sqrt{f_w} \tag{13}$$

Remember, for rough flow, $z_0 = k_r/30$ (i.e., $Re_r = k_r u_*/\nu > 3.3$) and for smooth flow, $z_0 = \nu/9u_*$ (i.e., $Re_r = k_r u_*/\nu < 3.3$). Of course for smooth flow, $3.3\nu/u_*$ replaces the roughness k_r.

Equation (12) and its approximation, Equation (13), are NOT empirical. They are an approximation (i.e., limit of $z_0\omega/\kappa u_{*wm} \to 0$) to the analytical solution of the equations of motion (1) using a time-invariant eddy viscosity.