Review of turbulent boundary layers

Let's begin with Navier-Stokes

$$\frac{D\vec{u}}{Dt} = \vec{F} - \frac{\nabla p}{\rho} + \nu\nabla^2\vec{u} \tag{1}$$

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

where $D\vec{u}/Dt$ is the 'total derivative', or 'material derivative'.

$$\frac{D\vec{u}}{Dt} = \frac{\partial\vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \tag{3}$$

For two-dimensional boundary layers, where fluid is moving in the *x*-direction, we neglect all terms in the *y*-direction. We also assume the pressure is everywhere hydrostatic. Finally, the boundary-layer approximation says –

L_x, $L_y >> L_z$ and $\partial^n/\partial x^n, \partial^n/\partial y^n << \partial^n/\partial z^n$

which leaves the conservation of momentum in the *x*-direction to be

$$\rho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z} \tag{4}$$

where τ is the shear stress

$$\tau = \rho \nu_t \frac{\partial u}{\partial z} \tag{5}$$

where ν_t is the ‘eddy viscosity’.

If the velocity varies slowly in time, the shear stress varies linearly in the vertical, with a maximum at the bed.

It turns out a convenient way to express the maximum value of the shear stress is by removing the dependence on the fluid density and casting it in terms of a velocity

$$u_* = \sqrt{\frac{\tau_b}{\rho}} \tag{6}$$

where τ_b is the shear stress at the bed and u_* is the shear velocity.

The eddy viscosity described in (5) can be described in any number of ways. It is NOT a property of the fluid, but rather the complicated mechanics embodied in the nonlinear terms of the Navier-Stokes equations. However, for boundary layers a simple model for it can be assumed.

$$\nu_t = \kappa u_* z \qquad (7)$$

where κ is the von-Karman constant (~0.4). Equation (7) is a result of the **Prandtl mixing-length hypothesis**.

Wave boundary layers

2D-velocity profiles (Grant and Madsen, 1979; 1986)

To proceed, we will assume that linear wave theory holds to the outside of the boundary layer. The boundary layer possesses a negligible thickness δ_w. This implies that shear stress is negligible for $z > \delta_w$ (remember, Airy theory necessarily assumes inviscid behavior), which simplifies (4) to become

$$-\frac{\partial p_\delta}{\partial x} = \rho \frac{\partial u_b}{\partial t} \qquad (8)$$

where u_b is wave orbital velocity and p_δ is the pressure at the top of the boundary layer.

Inside the boundary layer ($z < \delta_w$), pressure change is negligible in the vertical (i.e., the boundary layer is vanishingly small), so Equation (4) becomes

$$\frac{\partial(u_w - u_b)}{\partial t} = \frac{\partial(\tau_w/\rho)}{\partial z} \tag{9}$$

In reality, the shear stress in (9) varies with time. This means the eddy viscosity also changes with time. Despite these effects, Grant and Madsen (1979) assumed that the eddy viscosity will not vary with time. The resulting **time-invariant eddy viscosity model** is

$$\tau_w = \rho \kappa u_{*wm} z \frac{\partial u_w}{\partial z} \tag{10}$$

where u_{*wm} is the maximum shear velocity (i.e., the shear velocity associated with the maximum shear stress) and u_w is the velocity in the x-direction

Wiberg (1995) and others have shown that the time-invariant eddy viscosity model works well – particularly in the 'outer' boundary layer.

This assumption leaves

$$\frac{\partial(u_w - u_b)}{\partial t} = \frac{\partial}{\partial z}\left[\kappa u_{*wm} z \frac{\partial(u_w - u_b)}{\partial z}\right] \tag{11}$$

which can be solved knowing

$$u_w - u_b \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty \qquad \text{(i.e., } u_w \rightarrow u_b\text{)}$$

and

$u_w - u_b \rightarrow -u_b$ as $z \rightarrow 0$ (i.e., $u_w \rightarrow 0$)

Grant and Madsen (1979) used Kelvin functions to solve (11). Without dwelling on details, they propose a limiting solution for situations when both the wave boundary layer and roughness are small with respect to $\kappa u_{*wm}/\omega$.

$$u_w = \frac{2u_{bm}}{\pi}\ln(z/z_0)\sin\varphi\cos(\omega t + \varphi) \tag{12}$$

where u_{bm} is the maximum orbital velocity predicted from Airy theory, φ is the phase lead and z_0 is related to the effective roughness of the bottom.

For smooth-bed conditions (i.e., $Re_r = k_r u_*/\nu < 3.3$)

$$z_0 = \nu/9u_* \tag{13a}$$

For rough-bed conditions (i.e., $Re_r = k_r u_*/\nu > 3.3$)

$$z_0 = k_r/30 \tag{13b}$$

Where k_r is the ‘equivalent Nikuradse sand-grain roughness’ – in most cases, it can be approximated by the mean grain size in the bed.

Finally, the phase lead φ has the form

$$\tan\varphi = \frac{\pi/2}{\ln\left(\frac{\kappa u_{*wm}}{\omega z_0}\right) - 1.15} \tag{14}$$

The result is a logarithmic profile. Grant and Madsen (1979) and others have used this similarity to speculate upon the similarity of physical processes associated with turbulent energy production and sediment transport.

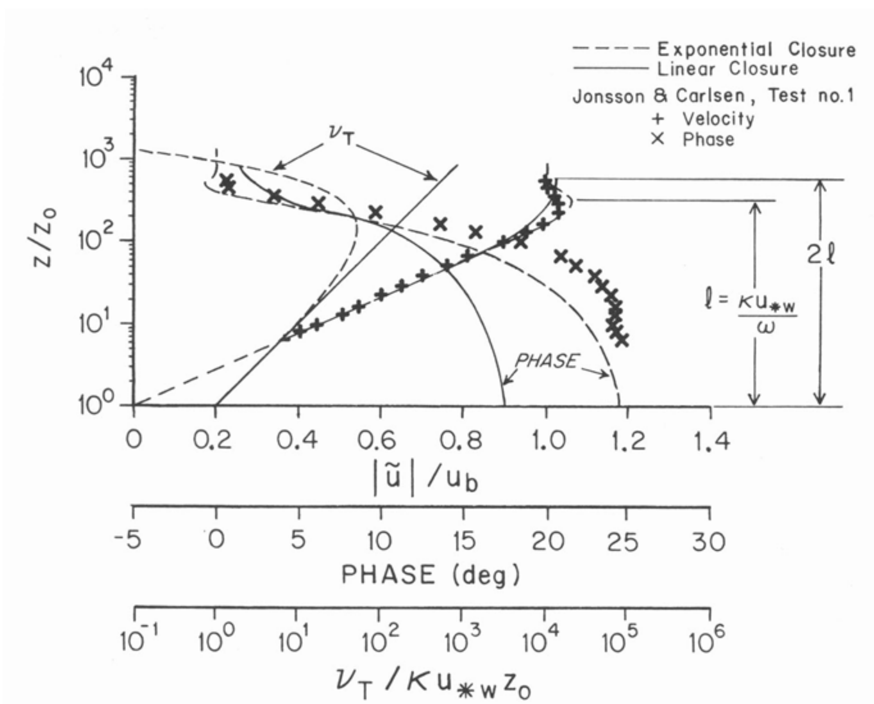

Figure obtained from Grant and Madsen (1986)

The ‘overshoot’ seen in the data and the models is ‘real’ and captured by both time-invariant models of the eddy viscosity.

Estimation of the shear stress

To estimate the maximum shear stress, you need to have some idea of the relation between the motions and the overall friction of the sea bottom.

Unlike the case in open channels (i.e., rivers), there aren’t any simple relations to estimate from the bulk-flow characteristics.

Remember the definition of the shear stress,

$$\tau_w = \kappa u_{*wm} z \frac{\partial u_w}{\partial z} \qquad (10)$$

Solving (10) and (12) simultaneously, we find

$$\tau_w = \tau_{wm} \cos(\omega t + \varphi)$$

where

$$\tau_{wm} = \frac{\rho \kappa u_{*wm} u_{bm}}{\sqrt{\left[\ln\left(\frac{\kappa u_{*wm}}{z_0 \omega}\right) - 1.15\right]^2 + \left(\frac{\pi}{2}\right)^2}} \qquad (15)$$

and

$$\tan\varphi = \frac{\pi/2}{\ln\left(\dfrac{\kappa u_{*wm}}{\omega z_0}\right) - 1.15} \qquad (14)$$

Wave friction factor

Because of the messy relationship between wave characteristics and the shear stress, a friction factor is often used to describe the relation of between the wave characteristics and the resulting shear stress. That relationship, first posed by Jonsson (1966), is

$$\tau_{wm} = f_w \rho u_{wm}^2 / 2 \qquad (16)$$

where f_w is the wave friction factor u_{wm} is the maximum wave orbital velocity.

Equations (16) and (15) can form an implicit equation for the wave friction factor in terms of the roughness at the bed.

That formulation is:

$$\frac{1}{4\sqrt{f_w}} + \log_{10}\left(\frac{1}{4\sqrt{f_w}}\right) = \log_{10}\left(\frac{A_{bm}}{k_n}\right) - 0.17 + 0.96\sqrt{f_w} \qquad (17)$$

Remember, for rough flow, $z_0 = k_r/30$ (i.e., $Re_r = k_r u_*/\nu > 3.3$) and for smooth flow, $z_0 = \nu/9u_*$ (i.e., $Re_r = k_r u_*/\nu < 3.3$). Of course for smooth flow, $3.3\nu/u_*$ replaces the roughness k_n.

For laminar boundary layers,

$$f_w = 2/\sqrt{Re_{bl}} \tag{18}$$

where $Re_{bl} = A_{bm}u_{bm}/\nu < 2000$.

You may also remember that the wave friction factor played an important in the dissipation of energy during shoaling.

Thickness of the wave boundary layer

The thickness of the wave boundary layer δ_w has taken on more importance as it has been found to regulate the depth of fluid muds. It is important to remember, however, the wave boundary layer equations of Grant-Madsen were derived for a Newtonian fluid – implicitly NOT the case in fluid-muds.

Despite Equation (12) being strictly applicable only near the bed, it suggests that the thickness of the boundary layer should occur when $|u_w| = u_{bm}$, or $z \approx \kappa u_{*wm}/(\pi\omega)$. This yields the common assumption –

$$\delta_w = \kappa u_{*wm}/\omega = \kappa A_{bm}\sqrt{f_w/2} \tag{19}$$

Equation (19) is the result reported in Grant and Madsen (1986) and commonly used to define the wave boundary layer thickness.

Algorithm to implement Grant-Madsen

In most situations where you want to find whether material will move on a margin, you will be interested in determining the wave friction factor f_w and the maximum shear stress τ_{wm}.

At the beginning of any enterprise, you will typically only have wave information and a bathymetric chart. In other words, you'll have H_s, T, and h. You'll also usually have some coarse knowledge of the substrate (i.e., the mean grain-size of the bottom).

Even if you have tripod data, you still will want to know f_w and τ_{wm}. In this case, you will probably already have u_{bm}.

So how do you proceed?

1 – Compute bottom orbital excursion amplitude from linear wave theory:

$$A_{bm} = \frac{H_{rms}}{2\sinh kh} = \frac{u_{bm}}{\omega}$$

Remember $H_{rms} = H_s / \sqrt{2}$.

2a – **If** $A_{bm} u_{bm} / \nu < 2000$, use Equation (18) to calculate f_w and then Equation (16) to calculate τ_{wm}. *You're done!*

2b – **If not**, assume rough turbulent flow and iteratively solve Equation (17).

3 – Obtain u_{*wm} from Equations (6) and (16).

4 – Check rough turbulent flow assumption by evaluating $k_n u_{*wm} / \nu$ in Equation (13).

5a – **If** $k_n u_{*wm} / \nu > 3.3$ **and** $A_{bm} u_{bm} / \nu > 2000$, then the flow is rough turbulent. Results from (2) and (3) constitute the solution. You should have τ_{wm} from Equation (16). *You're done!*

5b – **If** $k_n u_{*wm} / \nu < 3.3$ **and** $A_{bm} u_{bm} / \nu > 2000$, then the flow is smooth turbulent. Solve equation Equation (17) using $3.3 \nu / u_* = k_n$ and $z_0 = \nu / 9 u_*$.

6 – With f_w from Equation (17), recalculate τ_{wm} in Equation (16) and u_{*wm} in Equation (6).

7 – Use new value of u_{*wm} to iterate Equations (17), (16) and (6) until u_{*wm} does not change. *You're done!*

BBL mass transport and higher-order theory

As one might suspect, description of higher-order waves (more typical of nearshore regions where sediment transport and interaction with the bed is more important) is difficult.

Longuet-Higgins (1953) used an 'effective' time-variant eddy viscosity to derive his famous relationship for the net onshore transport near the bed.

Trowbridge and Madsen (1984) performed the second-order linear theory above with an explicit time-variant eddy viscosity. They found that the temporal variability was more important for higher order waves.

They also discovered that the net transport within the BBL was OPPOSITE of the direction of propagation of the waves. This is contrary most experiments and the original theory of Longuet-Higgins (1953).

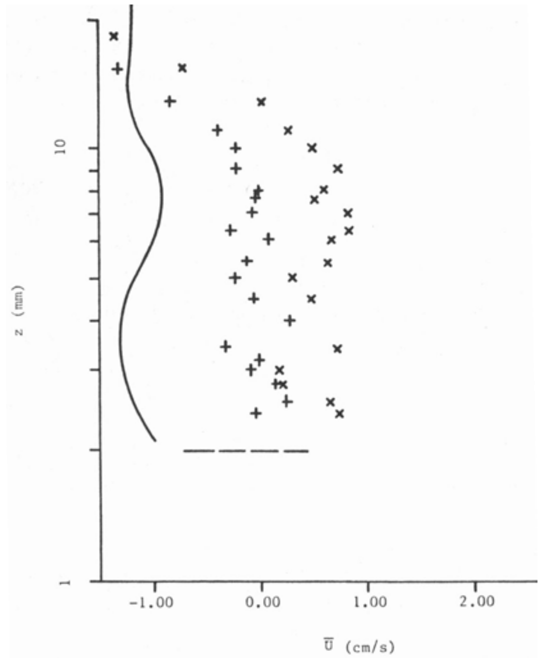

Figure obtained from Trowbridge and Madsen (1984)
using data from van Doorn (1981)