6. Dirac Comb and Flavors of Fourier Transforms

Consider a periodic function that comprises pulses of amplitude A and duration τ spaced a time T apart. We can define it over one period as

$$\begin{aligned} y(t) &= A, \quad -\tau/2 \le t \le \tau/2 \\ &= 0, \quad \text{elsewhere between } -T/2 \text{ and } T/2 \end{aligned} \tag{6-1}$$

The Fourier Series for y(t) is defined as

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{ik2\pi t}{T}\right) \tag{6-2}$$

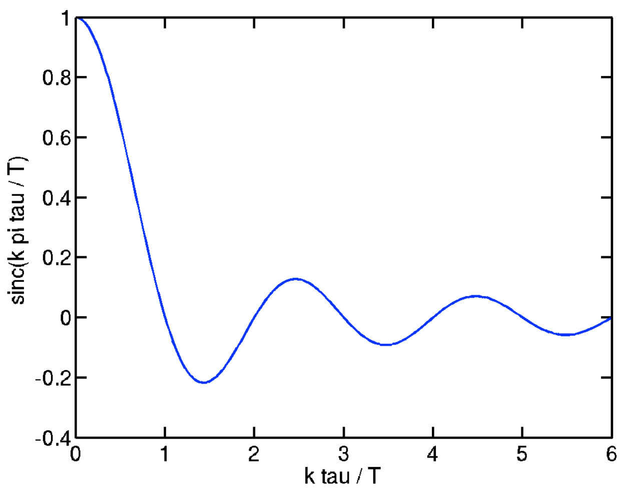
with

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\exp\left(\frac{-ik2\pi t}{T}\right)dt \tag{6-3}$$

Evaluating this integral gives

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\exp\left(\frac{-ik2\pi t}{T}\right)dt \\ &= \frac{A}{T}\left[\frac{T}{-ik2\pi}\exp\left(\frac{-ik2\pi t}{T}\right)\right]_{-\tau/2}^{\tau/2} \\ &= \frac{A}{T}\frac{T}{k\pi}\sin\left(\frac{k\pi\tau}{T}\right) \\ &= \frac{A\tau}{T}\frac{\sin\left(\frac{k\pi\tau}{T}\right)}{\frac{k\pi\tau}{T}} = \frac{A\tau}{T}\operatorname{sinc}\left(\frac{k\pi\tau}{T}\right) \end{aligned} \tag{6-4}$$

where we have used the relationship exp(iy) = cos(y) + i sin(y) to evaluate the integral with the cosine terms cancelling because of symmetry. The Fourier Series coefficients for a pulse train is given by a sinc function.



The largest amplitude terms in the Fourier series have $k < T/\tau$. Also if $T = \tau$ then the time series has a constant amplitude and all the coefficients except c_0 are equal to zero (the equivalent of the inverse Fourier transform of a Dirac delta function in frequency).

Now if we allow each pulse to become a delta function which can be written mathematically by letting $\tau \rightarrow 0$ with $A = 1/\tau$ which yields a simple result

$$c_k = \frac{1}{T}, \quad \lim \tau \rightarrow 0, \quad A = 1/\tau \qquad (6\text{-}5)$$

A row of delta functions in the time domain spaced apart by time T is represented by a row of delta functions in the frequency domain scaled by $1/T$ and spaced apart in frequency by $1/T$ (remember $f = k/T$).

Our row of equally spaced pulses is known as a Dirac comb. If we a define a Dirac comb in the time domain with the notation $C(t,T)$ such that

$$C(t,T) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \qquad (6\text{-}6)$$

then its Fourier transform is another Dirac comb function.

$$FT\left[C(t,T)\right] = \frac{1}{T} C\left(f, \frac{1}{T}\right) \qquad (6\text{-}7)$$

We can use the Dirac comb function in two ways.

Replication Operator

If we consider a continuous function $g_0(t)$ that is 0 everywhere except for $0 \leq t < T$ then convolution in the time domain with the a Dirac comb $C(t,T)$ replicates $g_0(t)$ to give a periodic function $g(t)$ with periodicity T (Figure 1).

$$g(t) = g_0(t) * C(t,T) \qquad (6\text{-}8)$$

Convolution in the time domain is multiplication in the frequency domain so we can write

$$G(f) = G_0(f).\mathrm{FT}\left[C(t;T)\right] = G_0(f).\frac{1}{T} C\left(f; \frac{1}{T}\right) \qquad (6\text{-}9)$$

The spectrum of the periodic function (Figure 1) is just a sampled version of the continuous spectrum of $g_0(t)$ with the samples scaled by a constant $1/T$. A periodic continuous function time has a discrete frequency spectrum.

Sampling Operator.

If we take our function $g_0(t)$ and multiply it by a Dirac comb $C(t,\Delta t)$ we obtain a sampled version $g_0(t)$ which we denote by $g_s(t)$ (Figure 2).

$$g_s(t) = g_0(t).C(t;\Delta t) \qquad (6\text{-}10)$$

Multiplication in the time domain is convolution in the frequency domain so we can write

$$G_s(f) = G_0(f) * \mathrm{FT}\left[C(t,\Delta t)\right] = \frac{1}{\Delta t} G_0(f) * C\left(f, \frac{1}{\Delta t}\right) \qquad (6\text{-}11)$$

The frequency spectrum of $G_s(f)$ is scaled by $1/\Delta t$ and replicated at intervals of $1/\Delta t$ (Figure 1). A discrete continuous function of time has a periodic frequency spectrum.

As we discussed in the lecture 5, we see again that sampling a time series removes our ability to discriminate between frequencies spaced at intervals of $1/\Delta t$. If we know our time series is limited to a maximum frequency

$$f_{\max} < \frac{1}{2\Delta t} = f_{Nyquist} \tag{6-12}$$

then there is no ambiguity in the frequency domain since the replicated spectra do not overlap. However is $f_{max} > f_{Nyquist}$ then the replicated spectra do overlap additively and we cannot discriminate between them in the frequency domain.

Recovery of a continuous time signal from a sampled time series

We have seen that if we sample at a frequency that is not at least twice the maximum frequency of the signal then we loose information. An interesting corollary of this is that if we sample a continuous signal that is band limited below the Nyqyist frequency then we can unambiguously recover the continuous signal from the sampled signal. This is known as Shannon's sampling theorem.

To show this, consider a repetitive spectrum that is obtained from a sampled time series and multiply it by a boxcar function of height Δt and width $1/\Delta t$. Provided the frequency content of the original signal is band-limited below the Nyquist frequency this gives us $G_0(f)$.

$$G_0(f) = G_S(f).B_c\left(f, -\tfrac{1}{2\Delta t}, \tfrac{1}{2\Delta t}\right)\Delta t \tag{6-13}$$

where $B_c(f,a,b)$ is a boxcar which has value of 1 for $a < f < b$ and 0 elsewhere.

In the time domain we can write

$$g_0(t) = \mathrm{FT}^{-1}\left[G_0(f)\right] = \left[\sum_{k=-\infty}^{\infty} g_0(t)\delta(t-k\Delta t)\right] * \left[\int_{-1/2\Delta t}^{1/2\Delta t} \Delta t \exp(i2\pi ft)\,df\right] \tag{6-14}$$

Now we can evaluate the integral

$$\int_{-1/2\Delta t}^{1/2\Delta t} \Delta t \exp(i2\pi ft)\,df = \left[\frac{\Delta t}{i2\pi f}\exp(i2\pi ft)\right]_{-1/2\Delta t}^{1/2\Delta t}$$

$$= \frac{\Delta t}{i2\pi f} 2\sin\left(\sin\left(\frac{\pi t}{\Delta t}\right)\right) = \frac{\sin\left(\frac{\pi t}{\Delta t}\right)}{\frac{\pi t}{\Delta t}} = \mathrm{sinc}\left(\frac{\pi t}{\Delta t}\right) \tag{6-15}$$

Equation (6-14) becomes

$$g_0(t) = \mathrm{FT}^{-1}\left[G_0(f)\right] = \left[\sum_{k=-\infty}^{\infty} g_0(t)\delta(t-k\Delta t)\right] * \mathrm{sinc}\left(\frac{\pi t}{\Delta t}\right) \tag{6-16}$$

Convolution is defined by the following

$$a(t)*b(t)=\int_{-\infty}^{\infty}a(\tau)b(t-\tau)d\tau \tag{6-17}$$

So we can write

$$g_0(t)=\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}g_0(\tau)\delta(\tau-k\Delta t)\text{sinc}\left(\frac{\pi(t-\tau)}{\Delta t}\right)d\tau \tag{6-18}$$

For each value of k the presence of the δ-function in the integral yields the value of the other functions at $k\Delta t$.

$$g_0(t)=\sum_{k=-\infty}^{\infty}g_0(k\Delta t)\text{sinc}\left(\frac{\pi(t-k\Delta t)}{\Delta t}\right)=\sum_{k=0}^{N-1}g_0(k\Delta t)\text{sinc}\left(\frac{\pi(t-k\Delta t)}{\Delta t}\right) \tag{6-19}$$

In the last term we have changed the limits of the sum to reflect the fact that $g_0(t)$ is only non-zero for N samples over the range $T = N\Delta t$.

We can see that when $t = j\Delta t$ with j an integer the above expression recovers $g_0(t)$ because the sinc term is zeros at all the other sample points. Elsewhere the value of $g_0(t)$ is obtained from a sum of multiple terms.

There are two implications. First, as noted at the start of this section, if the time series is band limited to $f_{\max} < 1/2\Delta t = f_{Nyquist}$ then we can recover the continuous time series from the discrete time series. The continuous time series is described completely by the discrete samples.

Second Equation (6-19) represents a method to interpolate uniformly spaced data. Since the maximum amplitude of the Sync function decrease away from zero we can get a reasonable interpolation by summing only a few terms for discrete samples around the time of interest.

Review of the Different Flavors of Transforms.

We have come across four types of transform, which we will review here:

1. *Continuous time* ⇔ *Continuous frequency*. Non-periodic continuous functions in time and frequency are related by the integral transform

$$\begin{aligned}G(f)&=\int_{-\infty}^{\infty}g(t)\exp(-i2\pi ft)dt\\ g(t)&==\int_{-\infty}^{\infty}G(f)\exp(i2\pi ft)df\end{aligned} \tag{6-20}$$

We can write the Fourier transform in terms of its amplitude and phase

$$G(f)=|G(f)|\exp\left[i\theta(f)\right] \tag{6-21}$$

where

$G(f)$ is the amplitude spectrum

$\theta(f)$ is the phase spectrum

and $|G(f)|^2$ is the energy or power spectrum.

For many applications we are more interested in the amplitude/power spectrum than the phase

2. *Continuous time* ⇔ *Discrete frequency*. Periodic continuous functions in time transform to Fourier Series that are discrete (but not periodic) in frequency.

$$
\begin{aligned}
G_j &= \frac{1}{T}\int_0^T g(t)\exp\left(-\frac{i2\pi jt}{T}\right)dt \\
g(t) &= \sum_{j=-\infty}^{\infty} G_j \exp\left(\frac{i2\pi jt}{T}\right)
\end{aligned}
\tag{6-22}
$$

3. *Discrete time* ⇔ *Continuous frequency*. From the symmetry of the Fourier transform pair we can infer functions that are periodic and continuous in frequency yield discrete (but not periodic) functions in time

$$
\begin{aligned}
G(f) &= \sum_{k=-\infty}^{\infty} g_k \exp(-i2\pi fk\Delta t) \\
g_k &= \Delta t \int_0^{1/\Delta t} G(f)\exp(i2\pi fk\Delta t)\,df
\end{aligned}
\tag{6-23}
$$

(note that the integral is taken over one period in the periodic frequency function).

4. *Discrete time* ⇔ *Discrete frequency*. When both the time and frequency functions are periodic, then they are both discrete. This yields the discrete Fourier transform (DFT)

$$
\begin{aligned}
G_k &= \frac{1}{N}\sum_{j=0}^{N-1} g_j \exp\left(-\frac{2\pi ikj}{N}\right) \\
g_j &= \sum_{k=0}^{N-1} G_k \exp\left(\frac{2\pi ikj}{N}\right)
\end{aligned}
\tag{6-24}
$$

or in alternately (in Matlab)

$$
\begin{aligned}
G_k &= \sum_{j=0}^{N-1} g_j \exp\left(-\frac{2\pi ikj}{N}\right) \\
g_j &= \frac{1}{N}\sum_{k=0}^{N-1} G_k \exp\left(\frac{2\pi ikj}{N}\right)
\end{aligned}
\tag{6-25}
$$

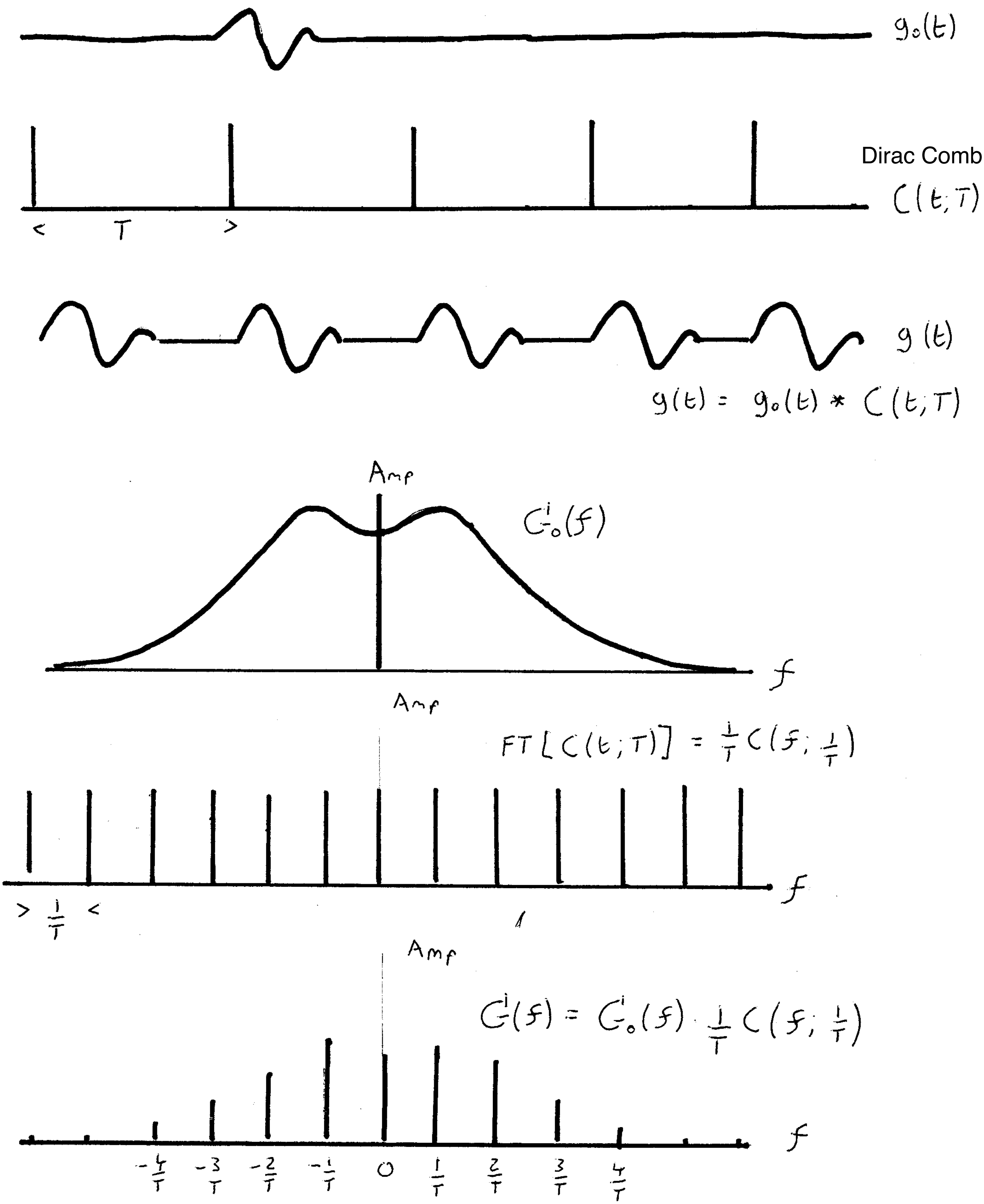


Figure 1 Dirac Comb as a replication operator

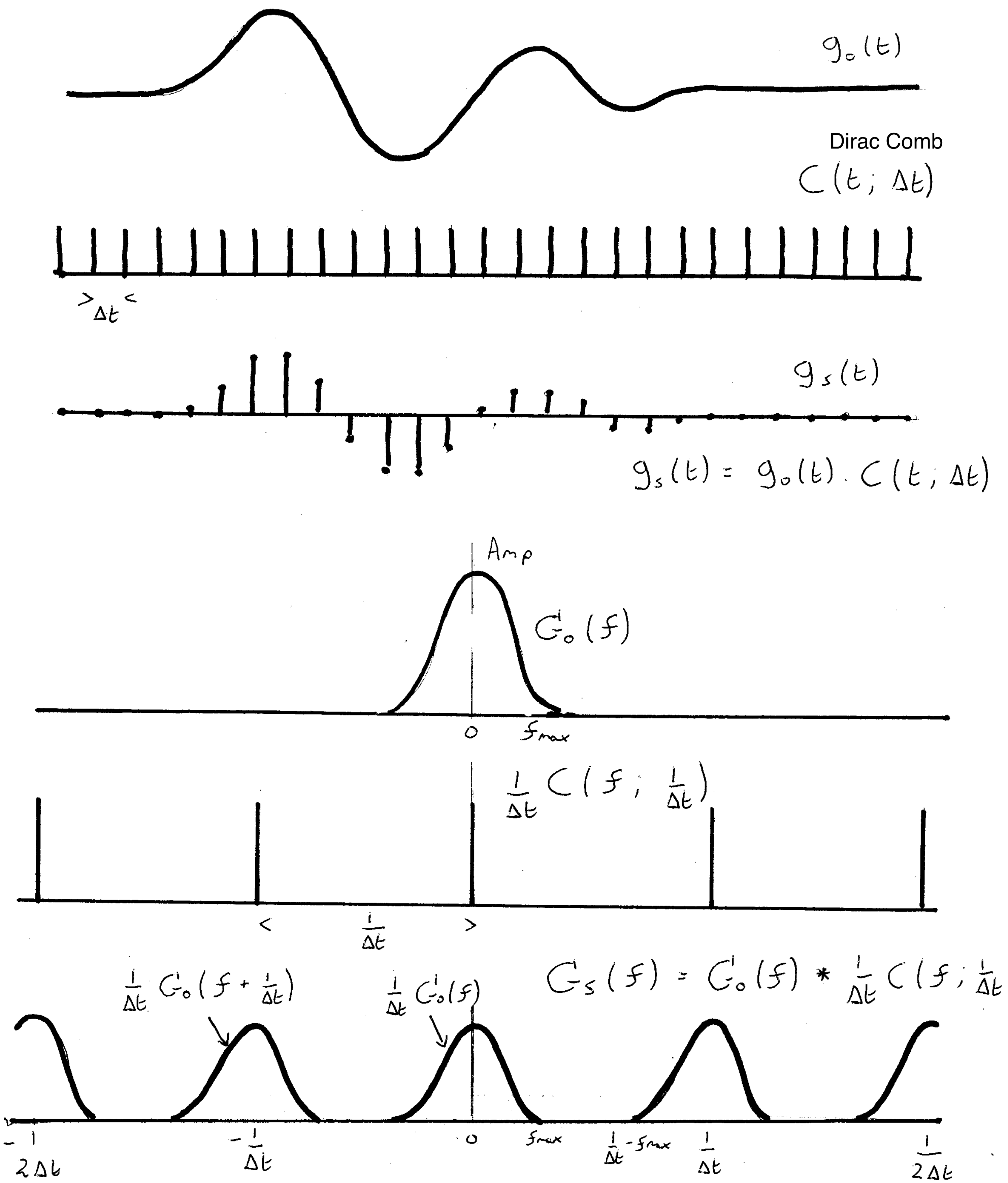


Figure 2. Dirac comb as a sampling operator