The Basic Sediment Transport Equations Made Ridiculously Simple

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1 Introduction

In sediment transport two important concepts are settling rate and boundary layer shear stress. Settling rate describes the tendency for sediment particles to fall out of suspension, and boundary layer shear stress describes the tendency for moving fluid to bring sediment particles into suspension (or at least get them moving along the bottom). So first we'll talk about Stokes settling, which is a simple way of describing sediment settling rate, then we'll talk about shear stresses in the boundary layer near a sediment bed. Finally, we'll talk about the Shields stress, which in a way brings both of these concepts together to predict when a moving fluid will transport sediment.

2 Stokes Settling

Stokes settling is a simple theory describing the velocity of a spherical particle settling through a fluid. It relies on a balance between the drag force which acts (in the upward direction) to slow the sphere down, and the gravitational force which acts (in the downward direction) to speed the sphere up. Because these two forces are balanced, the sphere is not accelerating. The drag force on the sphere is:

$$F_d = c_D \rho_f A_p w_s^2 / 2, \tag{1}$$

and the gravitational force is:

$$F_g = (\rho_p - \rho_f) V_p g. \tag{2}$$

Before we solve for w_s, we introduce R, the submerged specific gravity:

$$R = \frac{\rho_p - \rho_f}{\rho_f}, \tag{3}$$

and note that the drag coefficient can be defined in terms of the Reynolds number:

$$c_D = \frac{6}{Re}, \tag{4}$$

where:

$$Re = \frac{w_s D}{\nu}. \tag{5}$$

Inserting (3)–(5) into (1)–(2) and solving for w_s gives:

$$w_s = \frac{RgD^2}{18\nu} \tag{6}$$

This is the Stokes settling velocity, which gives us some idea of how fast sediment particles fall out of suspension, under some very limited circumstances.

3 Shear Stress in Boundary Layers

Now that we have a sense for the settling rates as a function of grain size, we need to get a handle on the shear stress τ in the boundary layer near a sediment bed, which will further help us understand the likelihood of sediment getting transported. There are two parts of a boundary layer that are of interest to us: the viscous sub-layer and the log layer.

3.1 Viscous Sub-Layer

In the viscous sub-layer, a very simple equation defines the shear stress on and near the sediment bed:

$$\tau = \mu \frac{du}{dz} \tag{7}$$

In this equation μ is the dynamic viscosity, which is a property of the fluid, and for water we know this property as a function of the temperature and pressure. $\frac{du}{dz}$ is the velocity gradient in this layer, but unfortunately, this layer is usually so tiny that in practice we can never measure the velocity gradient within it. The roughness of the sediment compounds the difficulty. So for figuring out the stress τ, this equation is pretty much useless.

3.2 Log Layer

The log layer is a different story. The equation for shear stress τ in the log layer is:

$$\tau = \rho_f \nu_e \frac{du}{dz}, \tag{8}$$

where ρ_f is the density of the fluid, and ν_e is the kinematic eddy viscosity. We know ρ_f as a function of temperature and pressure, and because the log layer is much thicker than the viscous layer, we can (usually) easily measure the velocity gradient within it. But what is the kinematic eddy viscosity? To get at this number we have to delve a bit into turbulence theory.

In turbulence theory, there is this thing called the "characteristic velocity scale within the turbulent flow", which is denoted u_*. Because there are lots of velocities within a turbulent flow, we have to define a "characteristic" velocity, which for our purposes can be thought of as the average velocity within the turbulent flow. (It is more complicated than this, but we don't really care.) Anyway, the people who study turbulence tell us that, in a turbulent boundary layer, we can know u_* as a function of the kinematic eddy viscosity and the velocity gradient in that layer:

$$u_*^2 = \nu_e \frac{du}{dz}. \tag{9}$$

If we plug (9) into (8), we get the shear stress as a function of u_*:

$$\tau = \rho_f u_*^2. \tag{10}$$

So that doesn't seem to help us much, because we still don't have any values for u_*. However, in river systems, people have found that:

$$u_* = \sqrt{ghS}, \tag{11}$$

where S is the slope of the river, and h is the depth of the river. So we can plug (11) into (10) and get the shear stress on the sediment bed in a river. This is cool. But in the ocean, things aren't that simple.

One way to proceed to obtain a more general solution is to use Prandtl's hypothesis, which relates u_* to the kinematic eddy viscosity:

$$\nu_e = \kappa u_* z. \tag{12}$$

Remember this is the equation that states that the scale of the turbulent motions are proportional to the height above the bed. In this equation, κ is VonKarman's constant, which is a universal constant having a value of 0.4.

With this equation we can go back to our original equations of stress in the log layer:

$$\tau = \rho_f \nu_e \frac{du}{dz} = \rho_f u_*^2, \tag{13}$$

and plug in Prandtl's equation to obtain:

$$\rho_f \kappa u_* z \frac{du}{dz} = \rho_f u_*^2. \tag{14}$$

We next integrate with respect to u and z:

$$\int du = \int \frac{u_*}{\kappa z} dz, \tag{15}$$

to obtain:

$$u = \frac{u_*}{\kappa} \ln z + C. \tag{16}$$

C is a constant of integration, which we let equal$-\ln z_0$, which give us:

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln\left(\frac{z}{z_0}\right). \tag{17}$$

z_0 is called the roughness length. With equation (17) we can make two (or more) measurements of velocity in the log layer and calculate the shear stress within that layer. By letting $z_0 = 30D$ (a common value for the roughness length), we only need to make one measurement of velocity in the log layer to get the shear stress on the sediment.

Special Hint: It is equation (17) that you must solve using the two measurements that you made during the laboratory exercise this week.

4 Shields Stress

With our understanding of settling rate and shear stress in the boundary layer, we can try to make predictions about the conditions under which sediment will be transported. For this we use the Shields stress and the particle Reynolds number. Since we know the shear stress in the boundary layer, we can calculate the Shields stress τ^* with the following equation:

$$\tau^* = \frac{\tau}{\rho_f R g D}, \tag{18}$$

and since we know u_* we can also calculate a quantity known as the particle Reynolds number:

$$Re_p = u_* D/\nu, \tag{19}$$

where ν is the kinematic viscosity (different than ν_e), which is a property of the fluid:

$$\nu = \frac{\mu}{\rho_f}. \tag{20}$$

For water, equations of state give us the value of ν as a function of fluid temperature and pressure.

When the fluid flow above a sediment bed is just high enough to start the substantial movement of particles, we call that the critical Shields stress, τ_c^*. Shields in some of his work compared τ_c^* to Re_p to come up with the Shields curve, which is an empirical curve generated from laboratory experiments. This curve gives us a rough idea of at what shear velocities sediment particles of a particular size will begin to move.

Special Hint: After calculating $u*$ and τ from your laboratory measurements, you should calculate τ_c^* and Re_p and plot the value you obtain on Shields' curve. How do your results compare?