Problem Set 1

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We will work on some of these problems in class. Turn in written solutions/discussion for 5 of the 8 problems below. You may discuss these with colleagues but solve as much possible on your own.

1. **Show** that a column of dry atmospheric air has internal energy $E$ and gravitational potential energy $PE$ with ratio

   $$\frac{E}{PE} = \frac{1}{\gamma - 1} \quad \text{where} \quad \gamma \equiv \frac{C_p}{C_v}$$

   For diatomic molecules dominating air, $\gamma = \frac{7}{5} = 1.4$, so $PE/E = 0.4$ as in the similar example worked out in class, which looked at small changes $\delta E/\delta P e$ when the air column was heated by a given amount $\delta^* Q \, C_p$ and $C_v$ are specific heat capacities at constant pressure and volume, respectively. $C_p = 1000 \, \text{JK}^{-1} \text{g}^{-1} \text{K}^{-1}$.

   In class we showed this for small changes in ideal gas heated in a box, with and without expansion allowed by a moveable lid (the weight of the lid being lifted was part of the potential energy $PE$ whereas here the atmosphere simply expands upward when heated... no lid).

   $$E = \int \rho C_v T dz; \quad PE = \int g \rho z dz$$

   with integrals extending over the whole height of the atmosphere from $z = 0$ up to where $p$ goes to zero.

   The equation of state $p = \rho RT$ and hydrostatic balance, $dp/dz = -g\rho$ can be used to relate the two integrals. Here we use $z = 0$ as the arbitrary reference for $PE$, and temperature $= 0^\circ \text{K}$ is the reference for $T$, where $p$ goes to zero.

   **Answer:** In order to understand the limit of integration, we can use the equation of state for dry air (ideal gas) and the hydrostatic relationship. This is using the equation of state, we write the hydrostatic relationship as

   $$\frac{dp}{dz} = -\frac{g}{RT} \rho$$
Integrating we get

\[ p(z) = p_0 e^{-\frac{g}{R}z} \]

Therefore, the integrals in the definition of the energies should go from zero up to infinity. Thus we have for the Internal Energy (E)

\[ E = C_v \int_0^\infty \rho T \, dz = \frac{C_v}{R} \int_0^\infty p \, dz \]

where the expression on the right comes from using the equation of state to write the definition of Internal energy in terms of pressure. The Potential Energy on the other hand is

\[ PE = \int_0^\infty \left( -\frac{dp}{dz} \right) z \, dz = \left( -pz \right) \Bigg|_0^\infty + \int_0^\infty p \, dz = \int_0^\infty p \, dz \]

The first term on the integration by part is zero since the pressure decays to zero faster that z goes to infinity. With the two energy expressions expressed above, the ratio \( E/PE \) is

\[ \frac{E}{PE} = \frac{C_v}{R} = \frac{C_v}{C_v - C_p} = \frac{1}{\gamma - 1} \]

(1)

2. We argued that the result of problem 1 is that for the climate system the atmosphere acts like the expandable working gas of a heat engine, while the ocean acts as a large ‘reservoir’ of heat.

Find the ratio of small changes in internal energy and potential energy, \( \delta E/\delta (PE) \), when a layer at the top of the ocean (is) heated by the sun and atmosphere. Treat the layer as a slab of uniform temperature \( T \) and thickness \( h \). Then

\[ \delta E = \rho h C_p \delta T, \quad \text{and} \quad \delta PE = \delta \int_0^h \rho g z \, dz = \delta \left( \rho \int_0^h g z \, dz \right) \]

(2)

where \( \rho \) is uniform in the layer. Note that as it is heated \( h \) increases but the mass \( \rho h \) per unit area is constant.

Here instead of an ideal gas we have \( C_p = 4000 \, J/\text{kg} \cdot \text{K}^{-1} \), and thermal expansion coefficient measured empirically: \( \alpha = -\rho^{-1} \partial \rho / \partial T = 2 \times 10^{-4} \, \text{sK}^{-1} \) (approximately, at temperature 20°C). In calculations like this it’s good to check that the units are the same in the two expressions for energy change (energy per unit surface area has dimensions of (mass)\times(velocity^2)/(length^2) = kg \, sec^{-1}, so \( C_v \) and \( C_p \) have dimensions sec^{-2}K^{-1} where K means degrees Kelvin).
**Answer:** As a first step, we can calculate the integrated potential energy for the slab layer. This is

\[
PE = \int_0^h \rho gzdz = \rho \int_0^h gzdz = \frac{1}{2} \rho gh^2
\]

Now, while the product \( \rho h \) is constant, the thickness does change when the layer is headed. This implies \( \delta (\rho h^2) = \rho h \delta h \). Therefore, we can write the small changes in potential energy for a slab layer heated from below as

\[
\delta PE = \frac{1}{2} g \rho h \delta h
\]

Therefore the ratio is

\[
\frac{\delta E}{\delta PE} = \frac{\rho h C_p \delta T}{(1/2) g \rho h \delta h} = \frac{2C_p \delta T}{g \delta h}
\]

Now, we want the result in terms on known quantities. From the conservation of mass, we have

\[
\delta (\rho h) = 0 \rightarrow \frac{\delta h}{h} = - \frac{\delta \rho}{\rho} = \alpha \delta T
\]

therefore, \( \delta h = \alpha h \delta T \), and we can write our solution

\[
\frac{\delta E}{\delta PE} = \frac{2C_p}{g \alpha h}
\]

Therefore, the ratio of internal energy and potential energy is inversely proportional to the thickness the surface ocean layer, where the incoming solar radiation warms the ocean.

3. **Compare** the Carnot heat engine with a heat engine whose cycle has two isobaric (constant pressure) and two adiabatic curves on the \( P - V \) (pressure-volume) plane (The Carnot Cycle instead is made up of two isothermal and two adiabatic curves).

We are heating (and expanding) at constant pressure, expanding adiabatically, cooling (and contracting) at constant pressure, the contracting adiabatically back to where we started. **What** is its efficiency (ratio of work done to heat input)? **Draw** the cycle on the pressure-volume diagram.

The motivation is to show explicitly the mechanical energy gain from the heat engine. This is a box of air with a moveable lid. A weight is placed on the lid, the air is heated and expands upwards at constant pressure. Then the weight is slid onto a shelf, and the gas expands further without heating (adiabatically, at constant entropy). The gas is then cooled under constant pressure and the lid drops back down. Another weight is slid onto the lid making it drop further, adiabatically. The lifting of weights is the mechanical energy output.
Answer:

• (1) - (2): **Heat input** in the system produces expansion at constant pressure. Temperature increases and so does the entropy (along an isobaric surface). There is work done by the gas.

In the P-V plane, since pressure is constant, the curve is just a flat line from \( V_1 \) to \( V_2 \). However, in the \( T-\eta \) plane, the relationship is a little bit more complicated. In order to find the functional relationship between \( T \) and \( \eta \), we take a look at the first law of thermodynamics. At constant pressure, this is just \( C_p dT = T d\eta \). We integrate this equation on both sides to get

\[
\frac{dT}{T} = \frac{\eta}{C_p} \quad \rightarrow \quad T(\eta) = T_1 e^{(\eta-\eta_1)/C_p}
\]

(5)

Therefore, in the \( T-\eta \) plane, we have exponential growth in temperature, as the entropy increases by the heating.

• (2) - (3): **Adiabatic expansion**. Entropy is constant. Temperature falls.

Here, in the \( T-\eta \) plane we have just a vertical line. However the curve in the P-V plane is not a line anymore. Therefore, looking at the first law for *adiabatic processes*, we have (see lecture notes)

\[
C_p dT = V dP \quad \rightarrow \quad T, P^\chi = T_1 P^\chi_1
\]

(6)

where \( \chi = (\gamma - 1)/\gamma \). Now, in terms of (specific) volume, we have

\[
PrV^\gamma = PV^\gamma \rightarrow \text{constant}
\]

Therefore, in an adiabatic process, the functional dependence between pressure and volume goes like \( P \sim 1/V^\gamma \). Therefore, the curve should show that.

• (3) - (4): **Heat taken out**. There is compression of the gas (at constant pressure). Temperature falls and so does the entropy (along an isobaric surface). Work is done to the gas. The functional dependence is similar to (1).

• (4) - (1): **Adiabatic compression**. Entropy is constant. Temperature rises. The functional dependence is similar to (2) (now pressure has to increase as volume decreases).

With \( d^*Q = T d\eta = C_p dT \), the heat in is \( C_p(T_2-T_1) \), heat out is \( C_p(T_3-T_4) \) so the work done is the difference \( C_p(T_2-T_1-(T_3-T_4)) \) and the efficiency is \( 1-(T_3-T_4)/(T_2-T_1) \).
The work can be calculated in different ways. In one way, work can be interpreted as the area enclosed by the p-v diagram; but also the T-S diagram area shows the heat input and output.

4. We remarked that it’s hard to warm your coffee by stirring it. Make a calculation demonstrating this, crudely estimating the temperature change caused by dissipating the kinetic energy of swirling coffee. The conversion from mechanical to thermal energy then leads to a warming proportional to the specific heat capacity at constant pressure, \( C_p = 4000 \text{ J kg}^{-1} \text{ K}^{-1} \).

**Answer:** There can be different approaches to this problem. The approach taken here is to assume that all of the kinetic energy that we place into our coffee, gets converted (via turbulent mixing) into internal energy. Then, we calculate the temperature change. Therefore

\[
\Delta E = \Delta KE \quad \rightarrow \quad C_p \Delta T = \frac{1}{2} |\langle u \rangle|^2
\]

where \( |\langle u \rangle|^2 \) is the mean (turbulent) kinetic energy of the 3D velocity in our coffee. Notice we don’t need to include the mass of our coffee, since it drops from the equation. With this, we get the change in temperature given our kinetic energy input

\[
\Delta T = \frac{|\langle u \rangle|^2}{2C_p} = \frac{\langle u \rangle^2}{8000} \quad (7)
\]

From the above result, we see that we would have to produce a (mean) velocity of about \( \langle u \rangle = \sqrt{8000} \approx 90 \text{ m/s} \) in order to warm our coffee by 1 degree Celsius. This is entirely a result of the heat capacity of water. Now, within our assumption we considered our system closed (no exchanges with our environment). If we considered heat loss into the air (which should increase as we increase our stirring), we would need then a greater stirring kinetic energy, which would again increase the heat loss into the air. Conclusion: indeed very hard to warm your coffee by stirring!

We discussed the way vorticity is involved in viscous dissipation of kinetic energy at a rate viscosity \( \times \) squared vorticity, so by simply observing the rate of decay of kinetic energy, and knowing that before viscosity takes effect, vorticity is increased in proportion to stretching of vortex lines, we have an estimate of this unbelievably complex turbulence process. Without this viscous damping the atmosphere and oceans might be much more energetic.
5. Why are steam engines useful? Consider the toy steam boat in the GFD lab. A small metal chamber about 1cm × 1cm × 0.5cm in volume was connected to two pipes that extended like twin exhaust pipes out the back, underwater. With water filling the chamber, a candle heated it until the water boiled. The big expansion of volume blew the liquid water in one of the pipes out the back, which sucked new cold water into the other pipe, refilling the chamber... and so on.

If the chamber and pipes were above the water surface, so that this became a dry-air heat engine, not involving water, would it have worked? Make some rough quantitative estimates. The heat absorbed by evaporating water (latent heat coefficient) is \(2.25 \times 10^6 \text{ J kg}^{-1}\) for water near boiling.

Without giving a full answer, the small volume of air would need to have a large temperature difference to do work; dry air would heat/cool more slowly than the boiling/condensing cycle.

6. A pot of 1 kg of water on a stove is heated from 20°C to 100°C with a steady 1 kW heating. How long does it take? Then it boils until the pot is dry. How long does that take?

**Answer:** For again, we assume a closed system (no heat loss to the air). Therefore all the heating goes into increasing the temperature of the water. This is

\[
dQ = (1000 \text{ J/s}) \Delta t = C_p \Delta T
\]

Here, \(\Delta T = 80^\circ \text{K}\), and \(C_p = 4000 \text{ J kg}^{-1} \text{K}^{-1}\). Therefore \(\Delta t = (4000 \text{ J kg}^{-1} \text{K}^{-1})(80K)/(1000 \text{J/s})\) (the mass here is 1 kg). This yields a time \(\Delta t = 320\) seconds, which is \(\Delta t \approx 5.30\) minutes to boil the kg of water.

In order to boil, we need to consider now the heat out, which is given entirely in the form of latent release. Also, we need to consider that once the temperature reaches boiling point, it doesn’t increase anymore (or it does, but very slowly, such that such increment is negligible and we can use a steady latent heat release), so that all the heat input from the stove is released by the latent heat. This implies

\[
Q_{in} = Q_{out} \quad \rightarrow \quad (1000 \text{ J/s}) \Delta t = (1kg)2.25 \times 10^6 \text{ J kg}^{-1}
\]

\[
\rightarrow \quad \Delta t = 2250 \text{ s} \approx 36.5 \text{ mins}
\]

Where we used a latent heat release value \(L_v = 2.26 \times 10^6 \text{ J kg}^{-1}\) for near boiling water (hence the assumption of \(\Delta T \approx 0\) for \(T = 100^\circ \text{C}\)).

7. The generation of mechanical energy by a heat engine is at most \((T_w - T_c)/T_w\) which is the efficiency of a Carnot cycle heat engine. \(T_w\) and \(T_c\) are the temperatures of the warm heat input and cold heat input reservoirs, respectively. Suppose the atmosphere has a temperature difference of 30°C between the tropics, where heat is injected and polar regions where heat is radiated to space (heat is radiated to space at all latitudes, but the heat engine works off the variation from the mean). Suppose also the heat flow through the system from tropics to polar regions is \(10 \times 10^{15}\) Watts (10 petaWatts).
What would be the rate of production of KE + PE of the atmosphere/ocean system if it had the efficiency of a Carnot cycle heat engine?

**Answer:** Consider the efficiency of the Carnot Cycle. This is

$$
\epsilon = \frac{(T_w - T_c)}{T_w} = 1 - \frac{T_c}{T_w}
$$

Since $\Delta T = 30^\circ$ K, then we can take a value of temperature at the tropics, say $20^\circ$ C = 293.73° K, which would lead to a temperature at the poles around $T_c = -10^\circ$ C, or $T_c = 263.73^\circ$ K. Therefore, the efficiency of this atmospheric Carnot cycle would be

$$
\epsilon = 1 - \frac{293.73}{263.73} = 1 - 0.898 \approx 0.1
$$

Therefore, we have an efficiency of 10%. Now, this means that only 10% of the heat input into our atmospheric system can be converted into mechanical work. That is

$$
\epsilon = \frac{\text{work}}{\text{Heat in}} = \frac{\text{rate of production of KE + PE}}{10 \times 10^{15}W} = 0.1
$$

(9)

Therefore, we have that the rate of production of $KE + PE = (10 \times 10^{15}W)(0.1) = 10 \times 10^{14} = 1 \text{ petaWatt}$. 

8. The ratio of PE/KE (potential energy to kinetic energy) is about 5 times for the atmosphere. **How much** energy would there be in the winds (and what would be their mean speed) if the dissipation time for kinetic energy is 5 days? The mass of the atmosphere is $5.3 \times 10^{18}$ kg.

**Answer:** If rates have the ratio $PE = 5KE$, then $PE + KE = 6KE = 10 \times 10^{14}W$. Therefore the rate $KE = 1.166 \times 10^{14}W$. The amount of energy in the winds is after 5 days of dissipation is simply rate of $KE\Delta t \approx 7.2 \times 10^{19} J$.

Now, from the definition of kinetic energy we can calculate the mean velocity of the winds. This is

$$
v = \sqrt{\frac{2KE\Delta t}{M_{atmos}}} = \sqrt{\left(\frac{2(7.2 \times 10^{19}J)}{5.3 \times 10^{18}kg}\right)} \approx 5.2m/s
$$

(10)
9. Thank you oceans. Levitus 2012 estimates that 93% of the extra heating of the Earth during the past 50 yrs of global warming has ended up in the oceans. The amount is estimated to be $2.5 \times 10^{23}$ Joules. If this extra thermal energy instead stayed in the atmosphere, how much would that warm the atmosphere of average? (In reality the atmosphere would radiate a lot of this excess heating away to space, but first assume it does not).

Answer: For this, we assume that all the heating gets converted into internal energy and our system is closed (no further losses). Then we have

\[ Q_{in} = M_{atmos}C_p \Delta T \]

With values for ideal gas (approximation of the atmosphere) $C_p = 1004 J/K$, $M_{atmos} = 5.3 \times 10^{18}$ and heating into our system is $Q_{in} = 2.5 \times 10^{23} J$, then we have

\[ \Delta T = \frac{Q_{in}}{C_pM_{atmos}} = \frac{2.5 \times 10^{23} J}{(5.3 \times 10^{18} kg)(1004 J/K)} \approx 47^o K \]

IPCC estimates of greenhouse gas induced global warming accounting for all the feedbacks in the system, are of order 1 Watt $m^{-2}$ extra heat gain, averaged over the Earth’s surface. This compared with the average solar energy flux of $\sim 1368$ Watts $m^{-2}$ in empty space above the Earth with 1/4 of that being the average incoming heat flux at the top of the atmosphere (432 Watts $m^{-2}$). An estimated 168 Watts $m^{-2}$ of solar radiation is absorbed at the surface (70% of which is ocean). With the tropics receiving much more that polar regions, the atmosphere/ocean circulation transports poleward thermal energy flux peaking at roughly $10 \times 10^{15}$ Watts (that is, 10 petaWatts) at 30° latitude ($\sim 5$ petaW of the two hemispheres).