The Earth’s radius, \(a\), is about 6371 km (for a sphere with the same volume as the actual Earth). The mean radius at the Equator is 6378 km, which is 21 km greater than the mean radius to the Poles (6357 km). This equatorial bulge arises from Earth’s rotation which is slowly decreasing as the Moon recedes. The rotation rate, \(\Omega = \frac{2\pi}{1 \text{ day}} \approx 7.292 \times 10^{-5} \text{ sec}^{-1}\). This means that typical horizontal velocity of the planet \(\Omega a = 465 \text{ m sec}^{-1}\). The typical acceleration of a point on the Equator is \(\Omega^2 a = 3.4 \times 10^{-3} \text{ m sec}^{-1}\). Not enough for us to feel but very large from the standpoint of a weather system or ocean current. Fluid motions may be affected by Earth’s rotation if their timescale \(T\) is significantly greater than \(\Omega^{-1}\) that is, 2 hours (after 2 hours or so you can see the change in direction of a swinging Foucault pendulum, as in the UW Physics/Astronomy building).

**Properties of air and water.** Air is mostly nitrogen (78%) and oxygen (21%), with inert argon at 0.9%. With fluorine, oxygen is the most electronegative of elements, attracting electrons to fill its outer shell: thus things oxidize. There are roughly \(2.7 \times 10^{25}\) molecules per cubic meter, with roughly \(10^{-9}\) m (=1 nanometer, nm) radius (this is 10 Angstroms, the Angstrom being defined as roughly the size of a hydrogen atom). The mean free path of air molecules, between collisions is 68 nm at sea-level, greater than their size but not that much greater.

Some of the properties that interest us include density, pressure, temperature, entropy, and many derived properties like sound speed (or ‘compressibility’), specific heat capacity, thermal expansion coefficient, freezing/boiling temperatures and latent heats of melting and evaporation.

An ideal gas has a ‘billiard-ball’ theory of elastic collisions that is remarkably accurate; see Gill for discussion of small corrections due to van der Waals forces. The equation of state for dry air is

\[
p = \rho RT
\]

(1)

\(p\) = pressure, \(\rho\) = density, \(T\) = temperature in (degrees) Kelvins, K, which are degrees Celsius + 273.15, \(R\) = gas constant for air = 287.04 J kg\(^{-1}\) K\(^{-1}\). Note that \(R\) varies from one gas to another, but \(R = R^*/\text{ms}\) where \(R^*\) is the universal gas constant 8314.36 J kmol\(^{-1}\) K\(^{-1}\), and \(\text{ms}\) is the molecular mass (28.966 kg kmol\(^{-1}\) averaging over the molecules of dry air).

But, in fact, what is the definition of temperature \(T\)? This simple question is often overlooked in text-books. We address it later in these notes.

Water’s equation of state is conditioned by the polar nature of the water molecule. Oxygen’s large electronegativity pulls the two electrons away from the hydrogen atoms making the covalent bound uneven, and giving the molecule an electric dipole moment. Nearby hydrogen molecules attract one another giving the liquid some long-range order. Water is a generally a remarkable substance with much research currently into its structure. The 104.5\(^{0}\) angle of the oxygen and hydrogen ‘triangle’ gives a hexagonal symmetry to ice crystals and the 6-pointed snow-flakes. In its solid form, apparently this angle is close enough to the 109.5\(^{0}\) angle in a pyramid-like tetrahedron that it
fits nicely. This is a relatively ‘open’ structure of hexagonal rings giving ice a lower mass density than liquid water (most substances are more dense when frozen).

The density of pure water, about 1000 kg m\(^{-3}\) (that is, 1 kg per liter or 1 gram per cm\(^3\)) decreases with temperature, but near freezing it has a density maximum (T=4C); colder water actually floats on top of warmer water, helping ice to form and float (ice covered ponds may have played an important role in evolution of aquatic animals who winter over beneath). Dissolved salts in ocean water reduce the freezing temperature and eliminate this density maximum, although the thermal expansion of seawater is very small near freezing. Gill has an appendix giving a good empirical equation of state for water as function of T, S and P. Matlab m-files are very good for this: visit http://woodshole.er.usgs.gov/operations/sea-mat The older CSIRO m-files have recently been improved by a new product, TEOS-10 at http://www.teos-10.org

Above: fresh water (zero salinity) density vs. temperature (at sea level pressure)

It is important to realize how small the density variations are in ocean water: in a typical water column the density will range from 1025 kg m\(^{-3}\) at the warm top to 1040 kg m\(^{-3}\) in the cold abyss. This is less than a 2% difference and, the potential density, that is the density corrected for adiabatic changes due to the large range of pressure, ranges typically 0.2%... 2 \times 10^{-3} ... from top to bottom.

We use linear approximations to the equation of state for ocean water in many cases (taking the slopes of the curves to get \(\partial \rho / \partial T\), \(\partial \rho / \partial S\), \(\partial \rho / \partial p\). The potential density, because of the nonlinear equation of state (the curvy shape of the plot) has to be referenced to a relatively nearby depth (or pressure). We often reference it to the sea surface (to atmospheric pressure) which works quite well for the top 1000m or so of ocean, but deeper water with its subtle density contrasts needs more careful treatment. The concept of ‘neutral density’ has been developed to help with this observational problem (Jackett & McDougall, J. Phys. Oceanography 1997).
As described by Gill, the density of air is typically 1.2 kg m\(^{-3}\) at sea-level (800 times less dense than water) decreasing with a scale height of about 8 km to zero. Pressure in the atmosphere has SI units force divided by area, 1 Pa = 1 Newton m\(^{-2}\). At sea level the pressure is mostly the weight of the column overhead, about 10,000 kg of air. Multiplying by g we get that sea-level pressure is about 10\(^5\) Pa which is 1 Bar. The venerable millibar, mb, has given may to the hectoPascal (hP), but both are 10\(^{-3}\) bar or 100 Pascals.

Pressure in the ocean, also nearly hydrostatic, is usually given in deciBars, units of 10\(^4\) N m\(^{-2}\) because the pressure increases almost 1 deciBar per meter of depth. You will often see oceanographic data plotted over, say 0-5000 dB rather than 0-5000m.

We will provide more discussion of the equations of state that matter so much, in future.

**Thermodynamics: basic ideas.** So much for the numbers. Now we look at the thermal energy equation. This relates to Gill Ch. 3 and Vallis Secs 1.5-1.6 but with some added ideas.

*1st Law of Thermodynamics.* This relates the internal, thermal energy of the fluid to inputs of heat and work, both of which change the internal energy. The work is due to compression or expansion, essentially pressure force X distance (of the fluid compressing or expanding), and is written as \(p \, dv\), where \(dv\), is specific volume, or 1/density \(\rho\). \(dE\) is change in internal energy and \(d*Q\) is heating (the * means that Q is not a state variable….not a quantity uniquely specified by the temperature and pressure: heat is not really a noun, but more of a verb!). This heating may be from radiation, or heat conduction, or turbulent mixing (followed by molecular conduction of heat).

\[
dE = d*Q - p \, dv\]

Entropy \(\eta\) was invented in part to provide a ‘heat-like’ variable that is a state-variable, and is full or mystery and philosophy. For present purposes we define it by

\[
d*Q = T \, d\eta
\]

so

\[
dE = Td\eta - p \, dv\]

but with the caveat that entropy increase is greater than or equal to \(d*Q/T\), and is ‘equal’ only for gentle changes that gradually change the equilibrium conditions (i.e., not a violent shock wave in a supersonic flow). Vallis adds a chemical potential term which is a ‘heat of solution’, to express chemically induced heating but this effect is usually small in GFD. What is much more important are contributions to \(d*Q\) from phase change: evaporation/condensation and thawing/freezing or water. These need careful treatment and are dominantly important in A and O.

We develop the 1st law for an ideal gas satisfying (1), so that \(d(pv_s) = RdT\).

\[
dE = Td\eta + v_s \, dp - d(pv_s)
\]

Now define the specific heats at constant volume \(C_v\) and at constant pressure \(C_p\):

\[
C_v = \frac{d*Q}{dT} = Td\eta/dT \text{ at constant volume } v_s.
\]

\[
C_p = \frac{d*Q}{dT} = Td\eta/dT \text{ at constant pressure } p.
\]

Define their ratio \(\gamma = C_p/C_v\).

For any kind of process with an ideal gas, \(C_v = dE/dT\) (since \(E = CvT\)) and \(C_p = dh/dT\) (since \(h = E + pv_s = CvT + RT = (Cv+R)T = C_pT\)).
For an ideal gas $C_v$ does not vary with temperature and vanishes at $T=0$, so
\[ E = C_v T. \]
The 1st law becomes
\[ T \, d\eta = (C_v + R) \, dT - v_s \, dp \quad (3) \]
In its first form (2) we relate changes in internal energy, density and heating and in the second from (3) we center on changes in entropy, temperature and pressure (which is more suitable to GFD).

Now, if an air parcel stays at constant pressure, its heating $T \, d\eta = (C_v + R) \, dT$ which must be just $C_p \, dT$ by definition. So this establishes a remarkable relationship:
\[ R = C_p - C_v \]
for a simple ideal gas, and
\[ T \, d\eta = C_p \, dT - v_s \, dp \quad (4) \]

Enthalpy is defined as $h = E + pv_s$ so
\[ dh = dE + d(pv_s) \]
\[ = C_p \, dT \quad \text{using } E = C_v T \text{ and } pv_s = RT \]
simply. Enthalpy and energy are twins: $dh = C_p \, dT$ and $dE = C_v \, dT$ regardless of the process (constant $v_s$ or constant $p$ or...). Enthalpy is what heating produces at constant pressure, whereas internal energy $E$ takes up all the heating at constant volume, $dv_s = 0$. We will see some consequences shortly.

For ocean water and non-ideal gases, the Maxwell relations (see Gill, Vallis) give more general versions of the 1st law than eqn. (3):
\[ T \, d\eta = C_p \, dT - T \, \left( \frac{\partial v_s}{\partial T} \right)_p \, dp \]
which should be generally valid. The partial derivative is taken at constant pressure. This aspect of thermodynamics really requires more study (as in reading a thermo-book like Sears Thermodynamics).

The specific heats $C_p$ and $C_v$ for a monatomic ideal gas are given by
\[ C_p = \frac{5}{2} R, \quad C_v = C_p - R = \frac{3}{2} R \]
This will again become clearer below. Air is mostly diatomic and so heat addition can lodge in a wider variety of molecular rotation and vibration. This raises the heat capacity (the heating required per degree of temperature rise) to $C_p = \frac{7}{2} R, C_v = \frac{5}{2} R$. The numbers are
\[ C_p = 1004.64 \, \text{J K}^{-1} \text{ kg}^{-1} \]
which is $1/4$ of the specific heat value for water, $C_p|_{\text{water}} \approx 4000 \, \text{J K}^{-1} \text{ kg}^{-1}$ (see Gill appendix p603). Since water is 800 times denser than air, the heat capacity per cubic meter of water is about 3200 times that of air. The oceans dominate the heat storage in the climate system (more than 85% of the ‘extra’ heat from global warming now resides in the oceans), as well as the storage of water, carbon and salts. For completeness note from Gill that a layer of ocean 2.5m deep has the same heat capacity as all the atmosphere above, and if all the water in the atmosphere were condensed out it would form a layer ~ 0.23 cm thick. The fact that on average about 1 m of seawater evaporates per year, it follows that water has a very short residence time in the atmosphere: the ratio of these numbers suggests about 8 days.

**Potential temperature and potential density.** With temperature so dependent on pressure it is useful to define a related quantity that is conserved in absence of external heating or mixing. That is, in adiabatic or isentropic processes. If the entropy $\eta$ is conserved (if $d^*Q = 0$) then the 1st law (4) is
\[ C_p \, dT = \frac{dp}{\rho} \]
\[ = RT/p \, dp \]
so
\[ dT/T = \gamma \, dp/p \]
where \( \kappa = \frac{R}{C_p} = \frac{\gamma - 1}{\gamma} \). Now imagine moving an air parcel in altitude from its current pressure to a reference pressure \( p_r \). Its temperature changes from \( T \) to \( T_r \), according to

\[
T_r = T \left( \frac{p_r}{p} \right)^{\kappa} \Rightarrow \theta = T \left( \frac{p_r}{p} \right)^{\kappa} \tag{5}
\]

where we simply relabeled \( T \) as \( \theta \), the potential temperature. The consequence is of course that \( \theta \) increases upward in a stable dry atmosphere unlike the typical 10°C per km cooling of the actual temperature \( T \). It is interesting that in the meridional plane \((y,z)\) isotherms \( T = \text{constant} \) slope oppositely to \( \theta \) contours.

Crucially important effects of moisture are described in Gill 3.8 and Vallis 2.9.3. We unfortunately don’t have time to describe moist processes adequately in GFD, but we will touch on them in brief descriptions of change of phase and latent heat.

\( \theta \) is conserved in an isentropic process so it must be related to the entropy. The relationship (Gill 3.7.3) in general is found by sitting at the reference pressure \( p_r \) where \( T = \theta \) and applying (4) (with \( dp = 0 \)). This gives

\[
\frac{d\eta}{d\theta} = \frac{C_p(p_r, \theta)}{\theta}
\]

and for an ideal gas \( C_p \) is a constant. In this case we integrate to

\[
\eta = C_p \ln\left(\frac{\theta}{\theta_r}\right) = \ln\left[\frac{\theta}{\theta_r} \frac{C_p}{p}ight] \tag{6}
\]

which is slightly different from Gill’s expression: we define here a reference potential temperature \( \theta_r \), so as to keep the argument of the logarithm dimensionless (Gill’s ‘const.’ on the righthand side does not make this clear). This choice arbitrarily sets a reference value of entropy. Note that eqn (6) can be rewritten to give \( \eta(T, p) \) using the definition (5) of potential temperature \( \theta \). The result is

\[
\eta = \ln \left[ \left( \frac{T}{T_r} \right)^{C_p} \left( \frac{p}{p_r} \right)^{-\kappa} \right]
\]

This helps us to understand how specifying two of the state variables (say, \( T \) and \( p \)) gives us all the others (\( \eta, \quad E = C_v T \) and \( \varrho = p / RT \)).

**Potential density** is the density a parcel would have if moved adiabatically (isentropically) to a common reference pressure. Obviously is closely related to potential temperature, simply

\[
\varrho_0 = \frac{p_r}{R \theta} \tag{Vallis 1.110}
\]

with the same pressure dependence that \( \theta \) has in eqn. 5.

**Alternative equations of state.** The five interesting variables in a dry ideal gas are \( p, \varrho, T \), \( \theta \), \( \eta \) and \( E \). Specifying two of these uniquely determines the thermodynamic state, given that we have an equation of state \((p, \varrho, T)\) and the 1st law (recall that we know \( E = C_v T \)). Because of the usefulness of entropy we want an eqn of state connecting \((p, \varrho, \eta)\). This is found by using eqns (5) and (6) in \( p = \varrho RT \) to climate \( T \) in favor of entropy \( \eta \). The result is

\[
p/p_r = \exp(\eta/C_v) \left( \frac{\varrho}{\varrho_r} \right)^{\gamma} \tag{7}
\]

where \( p_r = \varrho_r R \theta \), defines \( \varrho_r \), and again \( \gamma = C_p/C_v \). This particular form of the ideal gas eqn of state does not seem to be very well-known but it is extremely useful.

Notice that the usual eqn of state \((1)\) has pressure varying linearly with density for an isothermal process, \( T = \text{const.} \). Whereas, eqn \((7)\) has pressure varies like \( \varrho^\gamma \) for an isentropic process (ideal diatomic gas, \( \gamma = 7/5 \)). This is intuitively right: adiabatic compression does work on the fluid and
makes the pressure rise more rapidly than if heat is allowed to escape the air parcel as it must for an
isothermal process.

Now once again we can rewrite the eqn of state for variables \((p, q, \theta)\) using (6):

\[
\frac{p}{\rho_r} = \left[ \frac{\rho \theta}{\rho_r \theta_r} \right]^{\gamma}
\]

(I have not seen this equation before deriving it this week! Please verify). It is very compact and
nicely contrasts \(p = qRT\).

**Thermal expansion coefficient.** This is defined as

\[
\alpha = -\frac{1}{\rho} \frac{\partial \rho}{\partial T}
\]

For an ideal gas \(\alpha = 1/T\), simply. At room temperature, \(T = 300\text{K}\), \(\alpha = 3 \times 10^{-3}\). For water \(\alpha = (0\text{ to }3) \times 10^{-4}\) that is, at least an order of magnitude smaller than with air. We begin to suspect the
atmosphere is the ‘expansion fluid’ and ocean is the ‘heat reservoir’ of the giant heat engine of the
coupled climate system.

**Molecular derivation of eqn of state: defining temperature \(T\) and….the idea of flux.** So, what in fact is
‘temperature’, \(T\)? Historically it had to do with you feel on cold and warm days, and the
observation that various materials, particularly metals, expand when heated and contract when
cooled. Mercury, being a liquid metal is particularly useful. Its thermal expansion coefficient is \(2 \times 10^{-4}\text{K}^{-1}\), a little less than water. (Gasoline at nearly \(10^{-3}\text{K}^{-1}\) would make a great thermometer but there are reasons for not using it; alcohol is usually used in common thermometers.)

Surprisingly many basic textbooks in fluid dynamics and chemistry never define \(T\). So here we
derive the ideal gas equation of state and see what \(T\) and thermal energy actually are: thermal energy is
microscopic kinetic energy of molecules, for the simplest monatomic gas near room temperature. For more
complex molecules temperature also reflects their rotation and possibly vibration (quantum physics
gets involved particularly at high energies).

Consider the pressure, which we define accurately as the force per square meter exerted by a fluid
on a solid boundary. Now use, essentially, Archimedes’ idea that you can replace a solid boundary
by more fluid and the pressure will be the same. Consider the molecules of a dilute gas impacting
the boundary. They bounce off, altering their x-momentum \(mu\) to \(-mu\): a net change of \(2mu\). \(m\)
is the mass of a molecule. \(u\) is the x-component of vector velocity \(\mathbf{u}\). That change in momentum
exerts a force whose time-integral, or ‘impulse’ is \(2mu\). We can stand a small distance away from
the wall and do book-keeping on the flow of x-momentum across a plane \(x = x_0\). Each molecule
crossing this plane carries with it a momentum \(mu\), (suppose for the moment that the magnitude of \(u\)
is the same for all molecules); the rate at which particles cross the \(x = x_0\) to the right is number
density \(n\) (molecules per cubic meter) times the velocity \(u\) of those molecules with \(u>0\); an equal
number pass \(x_0\) to the left with \(u<0\). Thus

\[
\text{the flux of x-momentum in the x-direction} = \text{the average of } (nu \times mu) = nm \ u^2.
\]
With a distribution of molecular velocities, say with \( n(u)du \) molecules having x-velocity between \( u \) and \( u + du \), the average \( mnu^2 \) becomes \( m\int u^2 n(u)du \) which is also \( \rho \) times the average of \( u^2 \).

Remember that for a function \( f(x) \) the average value of \( x^n \) is \( \frac{\int x^n f(x)dx}{\int f(x)dx} \).

This is somewhat confusing: momentum flux looks like energy. Note that the factor of 2 does not appear because it is taken care of by the average of \( u^2 \) which accounts for molecules going in both directions. Of course we have done some time-averaging here rather than keep track of each impact. Now \( \rho m \) is just the mass density of the gas, \( \rho = \rho m \).

\[ p = \rho u^2 \]

where again we mean a time average. The air molecules however have 3 velocity components of which \( u \) is only one. On average \( u^2 = \frac{1}{3} |u|^2 \) where bold \( u \) is the vector velocity. So

\[ p = \frac{1}{3} \rho |u|^2 \]

Now define temperature \( T \) to be the constant times the average kinetic energy (KE) of one air molecule,

\[ \frac{1}{2} m u^2 = \frac{1}{2} kT \]

for each of the 3 degrees of freedom, \( u \) is the x-velocity component, say) hence the total KE is

\[ \frac{1}{2} m |u|^2 = \frac{3kT}{2} \]

This then gives

\[ p = \frac{1}{3} \rho \left( \frac{3kT}{m} \right) = \rho kT/m \]

\( k \) is known as Boltzmann’s constant, \( k = 1.38 \times 10^{-23} \) Joules/molecule °K. If we also define the gas constant as

\[ R = k/m \]

then we recover the ideal gas equation of state.

\[ p = \rho RT. \]

So the various ways for writing the equation of state are this, and

\[ pV = nRT \quad (n = \text{number of moles of gas, } V \text{ is total volume}) \]

and

\[ pV = NkT \quad (N = \text{number of molecules, } V \text{ is total volume}) \]

with \( R = kN/n = km \).

So you see, the equation of state is really a definition of what temperature is; definitions of \( p \) and \( \rho \) are clear enough and together they lead us to say that \( T = \) the average kinetic energy of air molecules, divided by the universal Boltzmann constant \( 3k/2 \).

And, we immediately know that the specific heat capacity \( C_v \) of this monatomic ideal gas is \( 3R/2 \);

the relationship between the heating and temperature change is given by \( dE/dT = d(1/2 m |u|^2)/dT = d(3kT/2)/dT = 3k/2 = 3mR/2 \). The units of these expressions are all Joules K\(^{-1} \) for a single molecule. The specific heat capacity \( \text{(expressed per kg of fluid)} \) at constant volume, is then \( 3R/2 \) Joules kg\(^{-1} \) K\(-1 \).

(In doing GFD it is always valuable to keep track of the units on both sides of an equation...they must agree.)
The full development of the kinetic molecular theory by Boltzmann and others is a triumph of science. In particular we see that the with kinetic energy of individual air molecules all being the same on average (as a result of multiple, elastic collisions), the speed of the lighter molecules will be greater than that of heavier molecules. Light molecules in the atmosphere (like helium) are so fast that they escape Earth’s gravity and are gone.

We also can immediately calculate the average speed of air molecules: oxygen with molecular weight 32 has an individual mass of \(32/6.02 \times 10^{26}\) and an average speed \((3kT/m)^{1/2} = 482\) m sec\(^{-1}\).

At this point it would be good to review the whole idea of the stress tensor in a fluid (see Bretherton notes). Pressure \(p\) is the average value of the symmetric part of the stress tensor; the rest of the stress tensor represents **viscous stress** having components both parallel to and normal to a reference plane in the fluid (examples: viscous shear stress in fluid in a pipe and viscous normal stress in a falling stream of honey).

**Sound speed and compressibility.** The equations for sound waves (see Bretherton notes) give a wave propagation speed

\[c_s^2 = (\partial p/\partial \rho |_n) = \gamma RT\]

for an ideal gas. For air this is 347 m sec\(^{-1}\). It is remarkable that the propagation speed is only slightly less than the speed of molecules (and makes one wonder about nonlinear effects). Isaac Newton predicted the sound-speed, but in one of his very few errors assumed the air to be isothermal rather than adiabatic as the wave passed by. In water the sound speed is (from empirical measurement) about 1500 m sec\(^{-1}\). Whales and navies use underwater sound to great advantage. We are not very interested in sound waves is GFD1 but compressibility of both air and water is significant in affecting the stratification.

**More about specific heats.** We can now appreciate how thermal energy is stored in fluids. A ‘billiard ball’ gas with monatomic atoms has only the kinetic energy of flight (at the low energy levels of GFD). Its \(C_v\) value is \(3R/2\). A diatomic atom can store energy also in rotation (with its angular momentum) of the atom pair, or in vibration of the pair. At room temperature, the energy level is too low to excite vibration (the chemical bond is a potential-energy well and vibrations need to be well above the bottom of the well). Rotation of the atoms (like a pair of planets) is however active and this means that it takes more heating to raise the temperature a given amount (since, as we have just seen, temperature is the kinetic energy of just 3 of the degrees of freedom of the molecule). Rotation provides 2 more degrees of freedom, each with energy \(kT/2\). (2, not 3, because rotation about the line joining the two atoms has no significant effect.) So we have \(C_v = 5R/2\) (and \(C_p = R + C_v = 7R/2\)). Measurements show in fact that as the temperature is raised from near absolute zero to 5000K, \(C_v\) starts off at 3R/2, transitions to 5R/2 near room temperature when the energy is high enough to excite rotation, and finally transitions again to 7R/2 (T rises above 1000K finally reaching the new plateau at ~ 5000K) when vibration of the molecule sets in. This all relates to quantum physics; Einstein in 1907 explained this, building on the work of Planck.

All this gives a pretty good theory of heat capacity for air, and helps us to understand the physics behind atmospheric thermodynamics.

**Phase change; latent heat.** About 1/2 of the poleward heat transport in the atmosphere is in the form of latent heat...water vapor that has evaporated particularly in the subtropical warm oceans and is carried poleward as part of the atmospheric MOC. Where condensation occurs, the heat is released and becomes dynamical. In the northern hemisphere this occurs largely in the great storm tracks of Atlantic and Pacific. The other 1/2 is in the form of dry static energy...the sensible heat...
that one feels… corrected for adiabatic compression. The heating by condensation can be seen approximately in the diagnosed mean wintertime heating from atmospheric reanalysis data from Held et al. J Climate 2002, is shown just below: a significant part of this pattern is latent heat release (in addition to convergence of sensible heat, radiation, and some transient eddy heat fluxes). Notice the strong heating in the storm northern storm tracks and tropics.

To evaporate water requires $2.5 \times 10^6$ Joules per kg at 30°C (or $2.25 \times 10^6$ J kg$^{-1}$ at 100°C). This is a massive amount of heat. To sense this compare the time it takes to heat water to boiling on a stove with the time it takes to boil the kettle dry (the latter is considerably longer).

**Clausius-Clapeyron equation.** Warm water evaporates much more rapidly than cold water. The water molecules are fighting with air molecules for space, and warm water has faster moving molecules. The analysis is in Gill 3.4, although he could have shown the simpler conclusion from eqn.3.4.8 that

$$e_w = e_w(T_0) \exp((T-273.15)(C_{pv} - C_{pw})(T_0^{-1} - T^{-1}))/2$$

where $e_w$ is the pressure exerted by water vapor at equilibrium for the temperature $T$ (in this equation $T$=273.15 is the Celcius temperature). $C_{pv} \approx 4R = 1148$ J kg$^{-1}$ K$^{-1}$ is the specific heat for water vapor and $C_{pw} \approx 4182$ kg$^{-1}$ K$^{-1}$ is the specific heat of liquid water. $T_0 = 273.15$ (the zero point of Celcius temperature). Thus we see in the climate system that in the tropics, the sun’s warming of the ocean yields an upward heat flux with latent heat larger than sensible heat, and vice versa at high latitude where the oceans are cold.
The figure below shows the water vapor pressure to be about 4 times as big at 30°C as at 5°C. With 50 mb of vapor pressure represents 5% of atmospheric pressure. The increase in $e_w$ due to global warming of the oceans is one of the most reliable predictions in climate science: the slope of the curve gives 7% increase in water vapor per degree of temperature rise; a doubling of CO2 may cause 3°C or so warming this century, and that will increase the water vapor pressure by about 21%! See Held & Soden J Climate 2006.

The figure below shows from Held & Soden, Ann Revs Energy Environ. 2000, the total water vapor in the air column, and closely related to it, the sea surface temperature. The upper panel showing the clear-sky greenhouse effect (the ‘blanket’ due to water vapor and other non-cloud tracers) also mirrors the water vapor. Notice the numbers: 50 kg m$^{-2}$ of invisible water in the air in the tropics. Comparing with the 10,000 kg of air in this column, we see an average specific humidity of 0.5% (or 5 grams water vapor per kg of air).
If this fig. is not in color, the tropical band has highest values: ~ 50 kg/m$^3$ (middle panel), ~300 K (lower panel), top of range (upper panel).

Remarks. Thermal energy is ‘rich’. If you doubt this try warming your coffee by stirring it. The numerical values of specific heat capacities and latent heats tell the story. The heat engine of the A and O convert internal thermal energy to mechanical energy, and eventually viscous dissipation takes away that mechanical energy again. We did not introduce the Carnot cycle of a heat engine, but it describes the cyclic conversion of heating into ‘work’ (that is, into generating kinetic and potential mechanical energy. Analyzing atmosphere and ocean using these ideas can be very fruitful, and yet
relatively few people do so. We did not talk about moist potential temperature despite the discussion of water vapor. To go farther we need to begin with dynamics of stratification and rotation, after reviewing the hydrostatic balance in a fluid at rest.

Hydrostatic balance: fluid at rest. See Gill 3.5. Balancing the vertical pressure gradient and gravity (true gravity corrected for rotational acceleration) gives
\[ \frac{\partial p}{\partial z} = -\rho g \]
where \( g = 9.8 \text{ m sec}^{-2} \) approximately. For an ideal gas \( \frac{p}{\rho} = RT \) so
\[ p^1 \frac{\partial p}{\partial z} = -g/RT \]
which for the choice of an isothermal atmosphere, \( T = \text{const.} \) gives
\[ p = p_a \exp(-gz/RT), \quad \rho = \rho_a \exp(-gz/RT), \]
The scale height, defined as \( H_s = \frac{\rho}{(\partial \rho/\partial z)} \), is \( RT/g \sim 8 \text{ km} \) for the troposphere. For the oceans, with their empirical equation of state, \( H_s \sim 200 \text{ km} \) is much larger.