\[ \mathbf{F} = \mathbf{ma} \text{ for a fluid} \]
\[ \mathbf{F} = \text{body forces} + \text{surface contact forces (pressure, viscous stress)} + \text{inertial forces due to} \]
\[ \text{accelerating frame of reference (rotating Earth).} \]
\[ \text{rotating planet reference frame and } \text{Coriolis’ theorem for the rate of change of a position} \]
\[ \text{vector} \]
\[ \text{the Earth’s geopotential surfaces which define the ‘horizontal’ and ‘horizon’.} \]
\[ \text{Inertial oscillations and steady forced flow with planet rotation; inertial radius} \]
\[ \text{pressure and fluid surface profile with a circular vortex, including Coriolis effects} \]
\[ \text{Rossby number} \]
\[ \text{‘stiffness’ of a fluid with strong Coriolis effects: Taylor-Proudman approximation} \]

**Coriolis’ theorem:** let \( \mathbf{x}_f \) be a position vector from the center of the Earth, as seen by an observer in a reference frame fixed to the ‘distant stars’, that is, an inertial reference frame. Let \( \mathbf{x}_r \) be the position vector as seen by an observer rotating with the Earth. Note that if the two vectors are identical at time \( t=0 \), then a small time \( \delta t \) later they will differ by a vector \( \mathbf{\hat{\Omega}} \times \mathbf{x}_r \delta t \) (think of this as the product of the radius \( r \) and the angle \( \delta \theta \): arc length of a circle = \( r \delta \theta \)). It follows that the rate of change of

\[ \frac{d\mathbf{x}_f}{dt} = \frac{d\mathbf{x}_r}{dt} + \mathbf{\hat{\Omega}} \times \mathbf{x}_r \]

apply twice:

\[ \frac{d^2\mathbf{x}_f}{dt^2} = (\frac{d}{dt} \frac{d\mathbf{x}_r}{dt} + \mathbf{\hat{\Omega}} \times ) (\frac{d\mathbf{x}_r}{dt} + \mathbf{\hat{\Omega}} \times \mathbf{x}_r) \]

\[ = \frac{d^2\mathbf{x}_r}{dt^2} + 2\mathbf{\hat{\Omega}} \times \frac{d\mathbf{x}_r}{dt} + \mathbf{\hat{\Omega}} \times \mathbf{\hat{\Omega}} \times \mathbf{x}_r \]

use the triple-cross-product rule on the final term:

\[ \mathbf{\hat{\Omega}} \times \mathbf{\hat{\Omega}} \times \mathbf{x}_r = \mathbf{\hat{\Omega}} (\mathbf{\hat{\Omega}} \cdot \mathbf{x}_r) - \mathbf{x}_r (\mathbf{\hat{\Omega}} \cdot \mathbf{\hat{\Omega}}) \]

\[ = -\mathbf{\Omega}^2 r_i \quad \mathbf{\Omega} \equiv |\mathbf{\hat{\Omega}}| \]

In terms of velocity we now have

\[ \frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{dt} + 2\mathbf{\hat{\Omega}} \times \mathbf{u}_r - \mathbf{\Omega}^2 r_i \]

where \( r_i = |\mathbf{x}_r| \cos(\text{latitude}) \) is the cylindrical radius from the rotation axis to the fluid particle.

There are two new terms, the Coriolis inertial force and the centrifugal (center fleeing) inertial force (final term). This follows from the inherent acceleration involved in motion along a circular path...centripetal (toward the center) acceleration. Thus both terms are accelerations which look like forces to a rotating observer, sitting on the Earth. We can write the centrifugal inertial force as the gradient of a potential function:

\[ \mathbf{\hat{\Omega}} \times \mathbf{\hat{\Omega}} \times \mathbf{x}_r = \nabla (\frac{1}{2} \mathbf{\Omega}^2 r_i^2) \]

**The geopotential surfaces.** The MOM equation becomes

\[ \frac{D\mathbf{u}}{Dt} + 2\mathbf{\hat{\Omega}} \times \mathbf{u} = -\nabla \rho - \nabla \Phi \quad 2.1 \]

where \( \Phi \) is the geopotential. Surfaces \( \Phi = \text{constant near the Earth’s surface are known as the ‘geoid’}. \)

\[ \Phi = \Phi_G - \frac{1}{2} \mathbf{\Omega}^2 r_i^2 \]

where \( \Phi_G \) is the potential field for the ‘true’ gravity force: the integral effect of all the mass of the Earth, including its fluid envelopes. Mapping gravity has become a great sport using orbiting satellites (most recently the GRACE twin satellites, one following in the same orbit, 20 km behind
the other, with precise range finding between the two satellites, to the order of microns (millionths of a meter distance). For our GFD accuracy we will use a point mass idealization,

$$\Phi_G = -\frac{GM}{r^2}$$

$r$ being the full spherical radius. $G$ is ‘big G’, the gravitational constant, $6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ and $M$ is the mass of the solid Earth, $5.97 \times 10^{24} \text{ kg}$. The two potentials combine to give what we commonly call ‘gravity’. At the Earth’s surface, this is

$$\Phi \approx g z \text{ (near the Earth’s surface)}$$

where

$$g = 9.8 \text{ m sec}^{-2}$$

(varying from 9.79 to 9.83 m sec$^{-2}$ with latitude) and $z$ is the vertical coordinate. But it is worth seeing the shape of $\Phi$ more accurately. There is an Equatorial bulge, the radius of the Earth at the Equator being about 6378 km, and at the Poles being 6357 km, for an ‘bulge’ of 21 km. Of course the solid Earth has complex shape, especially with continents floating high above the sea floor. It is described by a series of spherical harmonic functions, so there is more to it than a simple bulge.

The surface $\Phi = \text{constant}$ close to the mean Earth surface is shown in the slides accompanying these notes. On top of the basic ellipsoid is all the gravity variation due to the contrast between ocean floor and floating continents, due to gravity anomalies hidden beneath and due to the oceans and atmosphere themselves. The time-averaged ocean surface has a complex topography which mostly relates to the topographic mountains and valleys on the sea floor. Smith & Sandwell (see slides) have used radar altimeters on orbiting satellites to infer the seafloor topography with remarkable accuracy. This same ocean water surface topography is used for mapping ocean currents and oceanic heat storage also.

The momentum equation 2.1 has a useful alternative form. A useful vector identity is

$$2(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla \frac{1}{2} |\mathbf{u}|^2$$

$$\frac{\partial \mathbf{u}}{\partial t} + (2\tilde{\Omega} + \mathbf{\omega}) \times \mathbf{u} = -\frac{\nabla p}{\rho} - \nabla \frac{1}{2} |\mathbf{u}|^2 - \nabla \Phi$$

where $\mathbf{\omega} = \nabla \times \mathbf{u}$ is the vorticity. If $\rho$ is constant then the righthand side becomes the familiar Bernoulli function:

$$\frac{\partial \mathbf{u}}{\partial t} + (2\tilde{\Omega} + \mathbf{\omega}) \times \mathbf{u} = -\nabla \left( \frac{p}{\rho} + \Phi + \frac{1}{2} |\mathbf{u}|^2 \right)$$

This form of the MOM equation shows how the Coriolis effect is in fact a planetary vorticity, $2\tilde{\Omega}$ which adds to the relative vorticity vector, $\mathbf{\omega}$.

Inertial oscillations and steady circulation driven by a horizontal force. This simple problem illustrates the Coriolis force in its simplest form: a slab of fluid moving horizontally, all the fluid particles doing exactly the same thing. Equivalently this could be a single point mass moving under the influence of Coriolis forces. It is important that this horizontal plane is a geopotential surface (so it’s like a small piece of the Earth, where the curvature of $\Phi$ surfaces can be neglected). The horizontal MOM equations are

$$u_t - fv = -\frac{p_x}{\rho} + F$$
$$v_t + fu = -\frac{p_y}{\rho}$$

where $f = 2\tilde{\Omega}$. (When we move to sphere $f$ will be multiplied by sin(latitude)).
with \( u, v \) independent of \( x \) and \( y \). In this case we argue that it is consistent to have pressure \( p \) independent of \( x \) and \( y \), hence
\[
\begin{align*}
\dot{u} - f\dot{v} &= F \\
\dot{v} + fu &= 0
\end{align*}
\]
Suppose \( F \) is a constant force exerted in the \( x \)-direction, starting at time \( t=0 \). At that time \( u=0, v=0 \).
Differentiate the \( x \)-MOM equation with respect to \( t \), and use the \( y \)-MOM equation to give
\[
\dot{u} + f^2 u = 0
\]
This is a basic oscillator equation, and o.d.e. with solutions
\[
\begin{align*}
u &= A \sin ft + B \cos ft, \quad v = - \int f u \, dt = A \cos ft - B \sin ft
\end{align*}
\]
where \( A, B \) are constants. Thus the natural frequency of this system is \( f \), the Coriolis frequency. The corresponding period of oscillation, \( 2\pi/f \), is known as a ‘half-pendulum day’, or the period of a Foucault pendulum (such as the one hanging in the Physics Astronomy building at UW). This period ranges from 12 hours at the Poles to infinity at the Equator.

We don’t yet have the solution. Notice that in differentiating we lost the force \( F \), which is constant in time. Equations like these coupled o.d.e.’s, when driven by a forcing term on the righthand side, are often solved by dividing the solution into homogeneous and particular parts,
\[
u = u^H + u^P
\]
The homogeneous part is a solution with no forcing, \( F=0 \). Take this to be the solution found above. The particular part of the solution is any single solution of the forced problem. Here, with \( F=\text{constant} \), we can look for a steady (constant in time) particular solution:
\[
\begin{align*}
-f\dot{v}^P &= F \\
f\dot{u}^P &= 0
\end{align*}
\]
Together the two parts allow us to determine the unknown amplitudes:
\[
\begin{align*}
u(t=0) &= B = 0 \\
v(t=0) &= v^H(t=0) + v^P \\
&= A - F/f.
\end{align*}
\]
So \( A = F/f \), and the full solution is
\[
u = F/f \sin ft, \quad v = (F/f) (\cos ft - 1).
\]
The fluid oscillates in circles…inertial circles, and has a mean flow perpendicular to the force \( F \) (here ‘southward’ if \((x,y)\), \((u,v)\) are (east, north) coordinates and velocities. Particles follow cycloidal paths.

While this solution is very simple, it shows how the fluid at first does not sense that it is on a rotating planet: it accelerates in the direction of the force. But then gradually the flow is deflected to the right in the Northern Hemisphere, and after % period (about 3.5 hours in Seattle) it is moving at right-angles to the force. This gives the sense that the importance of Coriolis effects depends on the time-scale of the flow as well as other things. A cumulus cloud forming in the afternoon may not have any significant Coriolis response, yet if it ‘anvils’ and flows outward, up at the tropopause, that outflow may begin to deflect after just an hour or two.

Another general result visible here is that Coriolis effects limit the strength of the velocity. Notice that the amplitude of the solution varies like \( 1/f \).

If you integrate the \((u,v)\) velocities with respect to time you find the equation for fluid particle paths, \( X(t), Y(t) \):
\[
X = (U_0/f) \cos ft, \quad Y = (U_0/f) \sin ft - (F/f) t
\]
where we have written \( U_0 = F/f \) as the velocity amplitude. Thus the radius of the inertial circles is \( U_0/f \). This is a useful scale amplitude which occurs frequently: for example in the Gulf of Tehuantepec in Central America the easterly winds blow through mountain gaps driving a strong oceanic eddy. If we imagine this ‘shock’ to the ocean (like our suddenly applied force \( F \), above), the water might respond by starting an inertial oscillation, moving downwind but then veering to the
right. Calculate the inertial radius $U/f$, taking $U \sim 0.1 \text{ m sec}^{-1}$ and $f = 2\Omega \sin(\text{latitude})$: the ocean eddies that form seem to be substantially bigger in diameter than $U/f$ but they may be initiated in this way. The winds in the atmosphere jet through the mountain gaps and they too feel a Coriolis force which may be unbalanced by pressure initially; calculate the $U/f$ radius for the winds (note that the density difference between air and water does not affect the inertial radius. The winds shown in the figure below have a reference scale (note $50 \text{ m sec}^{-1} = 111 \text{ miles hr}^{-1} = 97 \text{ knots}$ (nautical miles per hour).

Three gaps in the Central American cordillera concentrate the easterly Trade winds into jets, which drive ocean eddies. The size of the eddies is consistent with the inertial radius $U/f$. (image from Prof. Billy Kessler, PMEL/UW).
Model response of ocean currents to Tehuantepec wind forcing. Over the first few days a downwind-current develops and veers to the 'right', then becoming an anticyclonic eddy: estimate the Rossby number of this eddy. Later on we see a second eddy, cyclonic, develop and the vortex pair then evolve and finally separate. This kind of event will become clearer when we study 'geostrophic adjustment'.

Do we see inertial oscillations in the oceans and atmosphere? Yes; in fact they account for nearly ½ the kinetic energy in the ocean, not counting surface waves. The upper ocean (the mixed layer, 10 to 100m deep) is particularly active with inertial oscillations driven by winds at the surface. More generally, interial oscillations are a limiting form of oceanic internal waves, whose frequencies lie between $f$ and the buoyancy frequency $N$. These propagate in all 3 dimensions, and are buoyancy oscillations in one extreme, inertial oscillations at the other. The end of the internal wave spectrum at frequency $f$ is the most energetic region. The figure flow is from the Netherlands website http://www.cleonis.nl/physics/phys256/inertial_oscillations.php
Inertial oscillations also exist in the atmosphere, though the strong mean winds make them more difficult to observe. As internal waves near the Coriolis frequency, they appear radiated from jet streams, and also in the atmospheric boundary layer at sunset; when the sun’s heating goes away, the boundary layer is ‘released’ from turbulent cloud convection, and this is a sort of initial value problem, akin to turning off the force $F$ in our problem above. The lower atmosphere begins an inertial oscillation which can last through the night.

Let’s explore two more basic flows:

1. **Forced flow in a zonal channel with walls.** Here we have equations (2.2) with boundary conditions $v = 0$ at two east-west walls, lying at $y = y_0$, $y = y_1$. With $F$ a constant force, independent of $x$ or $y$, it is natural to take $p$ independent of $x$. The equations are then

   \[
   u_t - fv = F \\
   v_r + fu = -p_y / \rho
   \]

   But if the fluid is a layer of constant density and height (with a rigid lid and rigid bottom) then MASS conservation tells us that $u_x + v_y + w_z = 0$, yet $w = 0$ and $u$ is independent of $x$, then $v_y = 0$, and the boundary conditions require $v = 0$ everywhere. Totally simple, all that happens is:

   \[
   u = ft, \quad p = \int pfu \ dy + m(z) = -p_f^2 ty + m(z)
   \]

   where $m(z)$ is the vertical, hydrostatic variation of the pressure. Basically Coriolis has no effect on the zonal velocity because the eastward flow can lean on the channel walls, resisting it. There is a northward decrease of pressure, whose gradient balances the Coriolis force, transmitting the wall effect into the fluid.

2. **Circular vortex: finding the pressure.** This is a bit more interesting. Consider a vortex in a uniform-density fluid. With circular streamlines, it is natural to use polar coordinates. We assume the velocity to be horizontal with no variation with $z$. Let $v$ be the azimuthal (round-and-round, or ‘swirl’) velocity component and $u$ the radial velocity component, with corresponding coordinates $\theta$ and $r$.

   \[
   \partial u / \partial t + u \partial u / \partial r + (v / r) \partial u / \partial \theta - v^2 / r - fv = -p_r / \rho \\
   \partial v / \partial t + u \partial v / \partial r + (v / r) \partial v / \partial \theta + uv / r + fu = -p_0 / \rho r
   \]

   but for a symmetrical vortex, $u = 0$ and there is no variation with $\theta$, so
\[-v^2/r - fv = -p_r/\rho\]
\[u = 0\]

Now we can choose a ‘shape’ for the vortex, the variation \(v(r)\). A familiar choice is \(v = \Gamma/2\pi r\) where \(\Gamma\) is the circulation; this is a point vortex, singular at the origin, with zero vorticity outside the origin \(r=0\). It has area integrated vorticity equal to \(\Gamma\), owing to the singular swirl at the origin. But let’s choose a much more realistic velocity profile,
\[v = V_0 \left(\frac{r}{L}\right) \exp\left(-\frac{r^2}{L^2}\right)\]
which is a ‘solid body’ core (rotating without shear in \(r \ll L\)) with exponential decay at large \(r\).

Using the radial MOM equation above we find a nice explicit solution for the pressure field.
\[p = \rho fV_0 L \left[ -\exp\left(-\frac{r^2}{L^2}\right) - \frac{1}{2} Ro \exp\left(-\frac{2r^2}{L^2}\right) \right]\]
where \(Ro\) is the Rossby Number, \(V_0/fL\). The pressure field is very interesting. For zero rotation (\(Ro = 0\)) it is a simple Gaussian curve with a ‘dip’ (low pressure) at the center. This dip occurs whether the vortex spins clockwise (anticyclonic in northern hemisphere) or anticlockwise (cyclonic in the N.H.). Conversely if \(Ro\) is very small, the pressure field extends out to larger radius, and changes sign with the sign of \(V_0\): cyclones have low pressure dips and anticyclones have high-pressure cores.

For a given velocity amplitude, the cyclone’s center pressure is more extreme than the anticyclone’s center pressure. Coriolis and centrifugal forces are adding in the cyclone, yet are opposing in the anticyclone. Notice that this vortex might be in ocean or atmosphere; except for a scale amplitude of \(\rho\), the form of the pressure depends only on the Rossby number \(Ro = V_0/fL\), insensitive to \(\rho\).

Free surface height. Plots of the pressure, below show this dependence. The vertical MOM balance is hydrostatic, so if there is a free upper surface to the fluid at \(z=\eta\)…a water surface…it will show the pressure field, \(\eta = (p-p_A)/\rho g\), assuming the atmosphere above to have uniform pressure \(p_A\).

The plots below show a vortex with radius \(L = 50\) km, \(f=1 \times 10^{-4}\) sec\(^{-1}\) and velocity amplitude \(\pm 0.1, 1, 10\) and \(100\) m sec\(^{-1}\), for which the Rossby numbers are \(\pm 0.02, 0.2, 2, 20\). Notice that vortices without Coriolis effects have central pressure (or fluid surface dip) that varies like \(V_0^2\) (as in Bernoulli) yet at low \(Ro\) (strong Coriolis) the central pressure and dip or rise of \(\eta\) vary like \(V_0\).
Angular momentum $\mathbf{r} \mathbf{v} \times \mathbf{u}$ and vorticity $\mathbf{\omega} \equiv \nabla \times \mathbf{u}$. If the fluid motion near a point of interest is analyzed locally (the velocity expanded in a Taylor series in $x, y, z$), one finds the strain and vorticity as combinations of velocity gradients. The vorticity (call the vertical vorticity $\zeta$) turns out, locally, to be proportional to the angular momentum of a small sphere of fluid initially centered at this point. However for finite time the sphere deforms and over finite distance the velocity behaves differently, so the two quantities are not the same. For our circular vortices notice that a point vortex has finite total vorticity (that is vorticity integrated over the whole fluid) and has infinite angular momentum. It has zero vorticity outside of the origin yet finite angular momentum there. In fact, the two are related by $\zeta = r^{-1}(r \mathbf{v})$; $r \mathbf{v} \mathbf{r}$ is the angular momentum density at a given $r$. So it is the gradient of angular momentum $r \mathbf{v}$ that relates to vorticity. A more general relation between angular momentum and vorticity is this vector identity:

$$r^2 \mathbf{\omega} = -2 \mathbf{\hat{r}} \times \mathbf{u} + \nabla \times (r^2 \mathbf{u}) \quad r \equiv |\mathbf{r}|$$

so

$$\int_A r^2 \mathbf{\omega} \, dA = -2 \int_A \mathbf{\hat{r}} \times \mathbf{u} \, dA + \oint_C r^2 \mathbf{u} \cdot d\ell$$

where $A$ is an area, $C$ is the contour around it and $d\ell$ is an element of arc length along the curve $C$. This is a rather complicated relationship between the two important quantities, but in special cases can be useful. Note that angular momentum requires an origin in space for the position vector; the ‘parallel axis theorem’ relates angular momenta calculated with two different origins. Vorticity, on the other hand is defined without any such additional origin. When meteorologists speak of angular momentum they almost always are referring to the angular momentum of the westerly/easterly winds with origin either at the center of the Earth (with scalar radius $r$) or the distance to the rotation axis, $r \cos \theta$.

It is interesting that the Earth day varies in length by about 1 millisecond over an annual cycle (the Earth slows down in northern winter). This is because the stronger westerly winds of northern winter take some angular momentum away from the solid Earth, and give it back in summer. The exchange mechanism is surface frictional wind stress, plus pressure forces on mountain slopes. (see Hide et al. Nature 1980).

**Geostrophic Balance: horizontal flow, uniform density** (see Vallis §2.8). In the vortex example above we see at small Rossby number, $Ro$, the solutions converge on flow in which pressure gradient and Coriolis force are nearly in balance, and centrifugal acceleration is negligible. Suppose for now (f-plane or ‘flat Earth’), $\vec{\Omega}$ is purely vertical. The density $\rho$ is constant so far. Then

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \vec{\Omega} \times \mathbf{u} = -\frac{\nabla p}{\rho} \quad \text{(horizontal)}$$

$$U / T, \frac{U^2}{L}, fU, \quad P / \rho$$

$$1 / fT, U / fL \quad 1, \quad P / \rho UfL$$

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = -\frac{p}{\rho} + g \quad \text{(vertical)}$$

The horizontal and vertical MOM balances are very distinct. The scale analysis of the horizontal balance is shown above, in terms of scale horizontal velocity $U$, length scale $L$ and time scale $T$. Pivoting about the Coriolis term, the temporal acceleration $\partial u / \partial t$ scales like $1 / fT$. The advective acceleration scales like $U / fL \equiv Ro$, the Rossby number. If both of these non-dimensional
parameters are small, \(<<1\), then our scale pressure \(P\) should be \(\rho U f\). As we saw above, if Coriolis effects are negligible, then instead \(P\) often scales like \(\rho U^2\) or \(\rho U L / T\).

For small Rossby number, \(\text{Ro} <<1\), we thus have the much reduced horizontal MOM balance, which is called \textit{geostrophic balance},

\[
2 \vec{\Omega} \times \vec{u} = -\frac{\nabla p}{\rho} \quad \text{(horizontal)}
\]

or

\[-f v = -\frac{p_x}{\rho}, \quad f u = -\frac{p_y}{\rho}\]

There is a great deal of ‘fall-out’ from this simplification of the dynamics.

• the pressure acts as a stream function for the horizontal velocity,

\((u,v) = (-\partial \psi / \partial y, \partial \psi / \partial x)\). Here the velocity is directed along streamlines, \(\psi = \text{constant}\).

The flow is \textit{horizontally non-divergent}, \(\partial u / \partial x + \partial v / \partial y = 0\) (automatically following from \(\partial w / \partial z = 0\) or from the existence of \(\psi\)).

• in this case the vorticity is vertical, given by \(\zeta \equiv \nabla^2 \psi\).

• if boundary conditions are given for \((u,v)\), there are an infinite number of solutions of these geostrophic balance equations...they are just a statement of balance between \(p\) and \((u,v)\).

• therefore, small terms neglected in MOM must be important.

• the curl of geostrophic MOM balance is degenerately zero; that is, \(\nabla \times (\nabla p)\) is identically zero. Therefore if we look at the \textit{vorticity balance} these ‘big’ terms will not be there. Indeed we often end up solving complete problems by fixing attention on the vorticity equation.

• a meridional overturning circulation cannot be geostrophic balance in the zonal MOM sense. This is because the x-integral of \(p x / \rho\) vanishes and hence the x-integral of the meridional velocity \(v\) must vanish if it is purely geostrophic. Of course meridional overturning is a key feature of atmosphere and oceans, and to balance the Coriolis force waves, eddies, instabilities all must exist which are not purely geostrophic. This is sobering when one sees the Hadley, Ferrel, and Brewer-Dobson meridional circulations in observations.

• the vorticity of the fluid circulation, \(\zeta\) scales like \(U / L\), and the vorticity of the resting fluid, as seen by a non-rotating observer, is \(f\). Their ratio, \(\zeta / f \sim U / f L \equiv \text{Ro}\), the Rossby number. Thus geostrophic flow with \(\text{Ro} << 1\) has a vorticity field which is (as seen by the inertial space observer) vertical and dominantly \(2 \vec{\Omega}\). This is known as \textit{planetary vorticity}.

• The consequence of the dominant planetary vortex lines is ‘Taylor-Proudman stiffness’, to be explored below. Basically, the planetary vortex lines express an absolute angular momentum which is very difficult to tip or stretch.

\textit{Taylor-Proudman stiffness of a rotating fluid.}

Coriolis forces appear in the horizontal MOM equations, normal to the rotation vector. The vertical MOM balance is independent of this. For ‘thin layers’ of fluid, with motions having \((H / L)^2 << 1\) where \(H\) is here the height scale of the fluid velocity and \(L\) is its lateral scale, the vertical balance is hydrostatic. Our three MOM equations are then

\[-f v = -\frac{\partial}{\partial x} \frac{\partial p}{\partial x}, \quad f u = -\frac{\partial}{\partial y} \frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial}{\partial z} \frac{\partial p}{\partial z} - g\]

Take the z-derivative of x-MOM and x-derivative of z-MOM, to find

\(f \partial u / \partial z = 0\),

and similarly with the y- and z-MOM equations give

\(f \partial v / \partial z = 0\).

These are known as the \textit{Taylor-Proudman approximation}, for small Rossby number, small \(1 / f T\), and in absence of density stratification. They express the ‘stiffness’ of a rotating fluid with respect to
tipping or stretching of vortex lines. Our discussion of the stretching of a ring of fluid, conserving its angular momentum, shows how much force has to be exerted to do so, when $Ro << 1$.

Now take MASS conservation,
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$ Differentiate with respect to $z$ to get
$$\frac{\partial^2 w}{\partial z^2} = 0$$ in view of the vanishing of $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$. Thus if the fluid column is stretched or squashed vertically, it happens in a linear fashion, with $w$ varying linearly with $z$. (Vallis’ text argues incorrectly that $\frac{\partial w}{\partial z} = 0$ from these equations, his eqn 2.201, yet he has it right later on).

What is the accuracy of Taylor-Proudman? It is not simply $Ro << 1$ although that helps. It is a nice exercise in scale analysis, and yet it requires more information: the vorticity analysis, yet to come, will tell us.

The good thing about GFD is that it is progressively deductive. That is, as we add more and more effects (rotation, stratification…) we don’t throw away results like the ‘vector-tracer’ property of the vorticity, $\vec{\omega}$ and vortex tubes, but instead we modify or apply them appropriately in the new situations. Kelvin’s circulation theorem, which is taught close to the beginning of fluid dynamics texts, remains at the heart of GFD through all of its sophisticated fluids, stratified, rotating, compressible…