Further remarks on rotating coordinates: Energy. We remarked that the geopotential \( \Phi \) is a potential energy (per unit mass; \( \rho \Phi \) is the potential energy per unit volume): notice the way it occurs in the MOM equation as one of the famous Bernoulli terms. The mechanical energy equation is found by forming the scalar product of velocity with the MOM equation. For a single fluid particle,

\[
m \frac{D\vec{u}}{Dt} = \vec{F} \quad (\text{fixed ref frame})
\]

\[
m \frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = \vec{F} \quad (\text{rotating ref frame})
\]

where the \( \vec{F} \) vectors are all the forces that might act on the particle due to gravity, pressure, friction, external forces and the \( \vec{F} \) forces in the rotating frame include also the centrifugal term. Note also that the \( \frac{D}{Dt} \) acceleration term is different in the two different frames (since the velocities differ).

When we form the mechanical energy equation notice that the Coriolis term vanishes…it does no work on the fluid particle because the \( 2\vec{\Omega} \times \vec{u} \) is perpendicular to \( \vec{u} \) so their scalar product vanishes. This is not to say that Earth’s rotation has no effect on the energy of the fluid, because it hugely affects the velocity and pressure fields and they affect energy. But it encourages us to think in the rotating reference frame. The result then is

\[
m \frac{D(KE_r)}{Dt} = \vec{F}_r \cdot \vec{u}_r \quad KE \equiv \frac{1}{2} |\vec{u}_r|^2 \quad (\text{rotating ref frame})
\]

The kinetic energy changes due to the scalar product of force and velocity. If we explicitly write the geopotential part of \( \vec{F} \), we have an additional term \( \rho \vec{u} \cdot \nabla \Phi \equiv g \rho w \) where \( w \) is the vertical velocity and as before \( g \) is the familiar acceleration due to true gravity plus the centrifugal force. Notice that if the fluid particle moves a small distance \( \delta \vec{X} = \vec{u}_r \delta t \) we have

\[
m \delta(KE_r) = \vec{F}_r \cdot \delta \vec{X}_r \quad (\text{rotating ref frame})
\]

That is, mechanical energy change is equal to force times displacement of the force.

This is a very useful result, which can be used to diagnose the energy ‘budget’ of a rotating flow. If the Rossby number \( Ro \) is small we have \( \delta u = f \delta Y \) hence \( \delta KE_r = u \delta u = uf \delta Y \) or \( \frac{1}{2} (\delta u)^2 = \frac{1}{2} f^2 (\delta Y)^2 \). The force involved can be diagnosed from the KE equation above. If we stick to a geopotential surface, the work against \( \Phi \) is zero, so it is simple.

Energy depends on the frame of reference, more so that the simple additive effect on momentum of the frame of reference. In the simplest example, two observers moving with differing velocities disagree about the kinetic energy of a mass point that they observe, and the difference depends upon the absolute velocity, not just the difference in velocities of their reference frames. For, if one observer moving with x-velocity equal to \( U_0 \) sees a mass moving with x-velocity \( u \), he/she gives it a kinetic energy value of \( \frac{1}{2} (U_0^2 + 2U_0u + u^2) \). Compare this with an observer moving with x-velocity equal to 0: his/her KE observation is \( \frac{1}{2} u^2 \). So the two disagree by an amount \( U_0u + U_0^2 \). This just says the same thing as the graph of \( u^2 \) on the y-axis against \( u \) on the x-axis: a parabola. A given velocity difference (x-axis) relates to a KE difference that varies with the position along the x-axis (that is, varies with \( U_0 \)). Thus it is not surprising that observers accelerating with respect to one another (rotating and non-rotating) really disagree about energies. A related note is that rate at which work is done by a force exerted on a fluid, \( \vec{F}_r \cdot \vec{u}_r \) depends on the observers reference velocity.
Thermal wind equation

We have not considered stratified fluids in these lectures yet, but this key result is so important and so connected with the Taylor-Proudman approximation that it is worth introducing. Taylor-Proudman tells us that rotation ‘stiffens’ the fluid along lines parallel with the rotation vector. In the f-plane (flat Earth) model, this means that columns of fluid move like vertical pillars. Clouds of dye, fully 3-dimensional, begin to form 2-dimensional ribbons. Viewed from above they are thin, viewed from the side they are tall.

Adding stratification changes only the vertical MOM equation, at small Rossby number: the horizontal MOM equations remain in geostrophic balance. The vertical MOM balance is independent of this. For ‘thin layers’ of fluid, with motions having \((H/L)^2 \ll 1\) where \(H\) is here the height scale of the fluid velocity and \(L\) is its lateral scale, the vertical balance is hydrostatic. Our three MOM equations are as before

\[
-fv = -\rho_0^{-1} \partial p/\partial x, \quad fu = -\rho_0^{-1} \partial p/\partial y, \quad 0 = -\rho^1 \partial p/\partial z - g.
\]

yet now the density \(\rho\) is no longer a constant, yet it is taken to be constant in the horizontal MOM equation. This is a strict form of the Boussinesq approximation, valid roughly for stratification that is weak enough that the scale height of the density profile, \(H_\rho \equiv \rho/(\partial \rho/\partial z)\) is much greater than the ‘height scale’ of the motion, \(H \equiv u/(\partial u/\partial z)\) for example. Compressibility is neglected here. Thermal wind for a more general fluid is very similar, but instead of horizontal derivatives \((\partial p/\partial x, \partial p/\partial y)|_z\) at fixed height \(z\), we take horizontal derivatives along a constant pressure surface, \(p\). Take the z-derivative of y-MOM and y-derivative of z-MOM, to find

\[
f \partial u/\partial z = g \rho_0^{-1} \partial \rho/\partial y, \quad f \partial v/\partial z = -g \rho_0^{-1} \partial \rho/\partial x.
\]

These are known as the thermal wind equations for small Rossby number, small \(1/fT\), for an unstratified fluid we recover the Taylor-Proudman result (Vallis 2.8.4). In vector form they are:

\[
\frac{f}{\rho} \frac{\partial \mathbf{u}}{\partial z} = \frac{\ddet{\mathbf{g}}}{\rho_0} \times \nabla \rho \tag{1}
\]

where \(\ddet{\mathbf{g}}\) is the net gravity acceleration (points downward); the right-hand screw rule is used to point to the cross-product of two vectors. Now the fluid allowed vertical shear: the horizontal velocity can vary in the vertical direction, and with it, \(w(z)\) is no longer a linear variation in \(z\).

There is a great deal to be said about thermal wind balance. Like geostrophic balance itself, it does not ‘solve problems’ giving the full fluid circulation and density fields driven by boundary conditions, but instead it gives a vitally important ‘balance’ relationship between, now, horizontal velocity and density fields. It also is intimately connected with vorticity, with waves, and with energetics as we shall see. Notice that thermal wind balance is between spatial derivatives of the interesting quantities, so that integrating vertically to get the horizontal velocity will require extra knowledge of the velocity at an end point of the integration (that is, there is an unknown ‘reference velocity’ or often ‘barotropic velocity’ which is invisible to the thermal wind equations).
How does one get some intuition about this balance, (1)? Most simply, we know that geostrophic flow occurs with \( u \) and \( v \) in balance with horizontal pressure gradient \( (p_x, p_y) \). If fluid varies from very-dense to less-dense along the \( x \)-axis, this pressure gradient will change in the \( z \)-direction. And, with it, the \( u,v \) velocities. Combining the hydrostatic rule (‘weight of fluid overhead produces the pressure difference in the vertical) with geostrophic balance provides intuition. In practice the signs are hard to remember, so it is worth remembering a familiar flow: say the Gulf Stream. Imagine it flowing eastward along the \( x \)-axis; its \( u \)-velocity increases with \( z \), toward the ocean surface. The fluid to the north is cold and dense, to the south it is warm and less dense. Therefore \( \partial \rho / \partial y > 0 \), determining the signs in equation (1).

We will work with the thermal wind equations in a preliminary way, and return to full stratified dynamics after some time with the single-layer, constant density fluid (Vallis Ch. 3).

**Unstratified, single-layer, shallow-water systems**

As in Vallis 3.1, a single-layer (‘barotropic’) model has a horizontal pressure gradient given by \( \nabla p / \rho = g \nabla \eta \), where \( \eta \) is the elevation of the upper surface. The horizontal MOM equations are thus

\[
\frac{Du}{Dt} - fv = -g \frac{\partial \eta}{\partial x} \\
\frac{Dv}{Dt} + fu = -g \frac{\partial \eta}{\partial y}
\]

The \( u,v \) horizontal velocities are independent of \( z \), by the hydrostatic balance (aided possibly by Taylor-Proudman Coriolis effects). Then MASS conservation becomes

\[
\frac{Dh}{Dt} + h \nabla \cdot \vec{u} = 0
\]

for a fluid layer of depth \( h \) (which may vary in \( x \) and \( y \)). Here the \( \nabla \cdot \) operator is horizontal only \((\partial u / \partial x + \partial v / \partial y)\). This MASS conservation equation follows from integrating \( \nabla \cdot \vec{u} = 0 \) with respect to \( z \) and applying boundary conditions connecting the \( w \)-velocity with the movement of the free surface or possibly with motion up or down a sloping bottom. For a flat, level bottom boundary, we simply have

\[
\frac{D\eta}{Dt} + (H + \eta) \nabla \cdot \vec{u} = 0
\]

where now \( H \) is the mean layer thickness. For small amplitude motions \( \eta / H << 1 \) and \( \eta / L << 1 \) where \( L \) is the horizontal scale of the motion, we simplify to the linear form,

\[
\frac{\partial \eta}{\partial t} + H \nabla \cdot \vec{u} = 0
\]

Bottom topography (variations of layer thickness due to a sloping bottom) can be included if one goes back and allows \( H=H(x,y) \) in (3), giving new terms \( u \partial H / \partial x + v \partial H / \partial y \).

**Layered models of a stratified fluid**

“\( 1 \frac{1}{2} \)” -layer model. In Vallis § 3.2.1 we see a variant on the one-layer shallow-water model, in which a thin upper layer floats on top of a much thicker, and denser lower layer. This gives the same equations as above, yet with a reduced gravity \( g' = g \Delta \rho / \rho \) where \( \Delta \rho \) is the density difference.
between the layers. This is a surprisingly good model in several situations in the upper ocean and lower atmosphere.

**Layered models.** A fluid built as a ‘parfait’ (a pretty multi-colored drink) of many layers can be a very good model of a stratified fluid. Each layer behaves as in the one-layer model above, yet the hydrostatic relation couples the layers together. This is described in Vallis §3.4, where we find that the horizontal pressure gradient in a given layer, \( \nabla_H p_2 \) say, is related to the pressure gradient in the next layer above, \( \nabla_H p_1 \) say, by

\[
\rho_1^{-1}(\nabla_H p_2 - \nabla_H p_1) = -g' \nabla_H \eta
\]

where \( g' = g(\rho_2 - \rho_1)/\rho_1 \) and \( \eta \) is now the interface elevation between the two layers.

The exact analog of the thermal equation for the layered density fluid is Margules’ Relation formed from the hydrostatic relation (2) along with the layered MOM equations;

\[
f(\tilde{u}_1 - \tilde{u}_2) = \tilde{k} \times g_1' \nabla_H \eta
\]

where \( \tilde{k} \) is a vertical unit vector. This equation helps greatly to understand thermal wind balance. Notice that the vector velocity difference between the two layers points along contours of the interface height field. In just the same way, the thermal wind balance equation shows that the difference in horizontal velocity between two nearby heights \( z \) and \( z + \delta z \) points along contours of constant density.

**Vorticity and potential vorticity (PV) for a layered fluid.**

In §3.6.1 Vallis (also Gill §7.2.1) derives the PV equation for a hydrostatic fluid made of layers, each with uniform density yet progressively denser as one moves down from layer to layer. The key equation is the vertical vorticity equation (‘vert VORT’), his equation 3.70. Within each layer,

\[
\frac{D\zeta_a}{Dt} \equiv \frac{\partial \zeta_a}{\partial t} + (\tilde{u} \cdot \nabla_H)\zeta_a = -\zeta_a (\nabla_H \cdot \tilde{u})
\]

Here the \( \tilde{u} \cdot \nabla_H \) term is purely horizontal and the \( \nabla_H \) operator is horizontal, \( (\partial/\partial x, \partial/\partial y) \). \( \zeta_a \) is ‘absolute vorticity, \( \zeta_a = \nabla \times \tilde{u} \mid z + f \equiv \zeta + f \). If you recall the exact vorticity equation, this is a little peculiar. The full vector vorticity equation is normally written

\[
\frac{D\tilde{\omega}}{Dt} = (\tilde{\omega} \cdot \nabla)\tilde{u}
\]

for a constant-density fluid without Coriolis effects. \( \tilde{\omega} \) is the vorticity vector. (If the fluid is compressible and/or stratified there is important modification to this, Vallis eqn. 4.16). The righthand side is described as ‘stretching and tipping of vortex lines’. Here we are focusing on the vertical vorticity equation, so that term becomes \( \zeta_a \partial w/\partial z \), picking out the vertical stretching, with no horizontal vorticity within the homogeneous-density, hydrostatically balanced layers.

You can derive the key linear vorticity equation more simply by cross-differentiating the linearized version of the x- and y- MOM equations (2) to give

\[
\frac{\partial \zeta}{\partial t} + f(u_x + v_y) = 0
\]

or

\[
(7)
\]
\[
\frac{\partial \zeta}{\partial t} = -f(u_x + v_y) = fw_z
\]
(The full nonlinear vertical VORT equation is
\[
\frac{D\zeta}{Dt} = -(f + \zeta')(u_x + v_y) = (f + \zeta)w_z
\].

Anyway, equation 3.70 of Vallis needs to be corrected.

The barotropic potential vorticity equation for a homogeneous layer (which may be one of the layers of a density-layered fluid) is (Vallis 3.78)
\[
\frac{D}{Dt}\left(\zeta + f\right) = 0
\]
including the full nonlinearity of horizontal advection: \(D(\zeta)/Dt = \partial(\zeta)/\partial t + u\partial(\zeta)/\partial x + v\partial(\zeta)/\partial y\).

The key step is in writing the vertical stretching term \(\partial w/\partial z\) as \(\frac{(w_{\text{top}} - w_{\text{bottom}})}{h} \equiv h^{-1}Dh/Dt\).

This works because \(w\) is a linear function of \(z\), as we showed earlier in these notes. Therefore
\[
\frac{D}{Dt}\left(\zeta + f\right) + h^{-1}\frac{D}{Dt}\left(\zeta + f\right) = 0
\]
\[
= -\frac{(\zeta + f)}{h^2}Dh/Dt + h^{-1}\frac{D}{Dt}\left(\zeta + f\right)
\]
\[
= \left[-(\zeta + f)\partial w/\partial z + \frac{D}{Dt}\left(\zeta + f\right)\right]/h = 0
\]
showing agreement with our vertical vorticity equation.

**Long, hydrostatic water waves with rotation.**

Hydrostatic (long compared to the thickness of the fluid layer) waves come in two flavors: high frequency (greater than \(f\)) and low frequency (below \(f\)). Here on an \(f\)-plane we have no spherical-Earth effects and hence no Rossby waves. Without rotation, the MOM equations combine simply to give the hydrostatic gravity wave equation
\[
\frac{\partial^2 \eta}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial \chi^2} = 0
\]  
here and below, \(\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2\) (Vallis 3.93). This is exactly the ‘classic’ wave equation of mathematical physics. Look for a general wavy solution, \(\eta = \text{Re}(\tilde{\eta}\exp(i\tilde{k} \cdot \tilde{x} - i\omega t))\). The equation is homogeneous and so the ‘solvability condition’ for it is the dispersion relation, found by substituting back in the equation:
\[
\omega = \pm ck \quad k \equiv \tilde{k}, \quad c^2 = gH
\]
Thus frequency is simply proportional to wavenumber \(k\). There is no ‘pebble-in-the-pond’ effect of a narrow initial disturbance (pebble) making a long train of sine waves. For propagation in one dimension (\(x\)-) only, the more general solution is based on a change of variable to \(\tilde{\xi} = x + ct, \tilde{\chi} = x - ct\)
These are called characteristic curves of the hyperbolic wave equation. If you go through the transformation, the equation becomes simply
\[
\frac{\partial^2 \eta}{\partial \tilde{\xi} \partial \tilde{\chi}} = 0
\]
with solutions \(\eta = \frac{1}{2} (F(x-ct) + F(x+ct))\). Each of the two signals propagates without change of form, one to the right, the other to the left. Initial conditions at time \(t=0\) can be satisfied by choice of \(F\). (in general this 2d order p.d.e. needs another initial condition, leading to two more terms in the general solution, but this brief sketch gives you the solution for problems with zero initial fluid velocity and some specified initial fluid surface height profile at \(t=0\).)

**Effect of rotation.**

The generalization of equation (8) to include Coriolis effects is almost
\[
\eta_{\omega} - gH\nabla^2 \eta + f^2 \eta = 0
\]  

Vallis’ equations 3.98 set the stage for this. However there is something missing, of crucial importance (which is why he doesn’t show you this equation). In order to combine the MOM and MASS equations, the linearized versions of our equations (2) and (3), into a single wave equation we need to do a little magic. Gill’s text describes this particularly well (§§7.2,7.3) When you cross-differentiate the linearized MOM equations (∂/∂y of x-MOM minus ∂/∂x of y-MOM) and take a ∂/∂t MASS you get

\[ \eta_t - gH V^2 \eta = -fH \zeta \quad \zeta = v_x - u_y \]  

(10)

instead of zero on the righthand side. Vorticity pops up! So unless we have some reason to say the vertical vorticity is zero, we have to think further. Well, suppose \( \zeta = 0 \). Then we have wavy solutions as before, yet substituting the plane wave general solution yields the dispersion relation

\[ \eta = \text{Re}(A \exp(ikx + i\gamma - i\omega t)); \quad \omega^2 = f^2 + gH (k^2 + l^2) \]

Suddenly the waves have become dispersive! All these solutions, known as Sverdrup-Poincare waves have frequencies greater than \( f \), and the dispersion relation is not a straight line (or cone) as with \( f = 0 \). The group velocity,

\[ \bar{c}_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right) = \frac{gH (k,l)}{(f^2 + gHk \cdot l)^{1/2}} \]

is the \textit{barotropic Rossby deformation radius}, a horizontal length scale that expresses the relative importance of gravity and Coriolis effects. The version of \( \lambda \) for a stratified fluid will become our most important length scale in GFD. These are hydrostatic waves with \( u \) and \( v \) independent of \( z \). The longest waves become simple inertial oscillations with frequency \( f \), identical with our calculation for a particle on an \( f \)-plane: here it is a slab of fluid on an \( f \)-plane. For shorter wavelength horizontal propagation occurs, with group velocity that limits on \( c \). Of course if we continue to even shorter waves, \( (kH) > 1 \) (that is, \( H/L > 1 \)) we will see short, non-hydrostatic gravity waves. The dispersion relation then limits on \( \omega = \frac{g}{|k|^{1/2}} \)

Yet we had to neglect the righthand side of (10) to derive these waves. Clearly (10) is telling us that vorticity \( \zeta \) and height fields \( \eta \) are coupled together by Coriolis effects. So, we need to use the vorticity equation. It’s linearized version is equation (7, lower). It can be rewritten

\[ \frac{\partial \zeta}{\partial t} = \frac{f}{H} \frac{\partial \eta}{\partial t} \]

because \( w \) varies linearly in \( z \), and is equal to \( \partial \eta / \partial t \) at the top of the fluid. Integrate in time to give

\[ \zeta = \zeta_0 + \left( \frac{f}{H} \right) (\eta - \eta_0) \]

where the zero subscripts indicate the value at the initial time, \( t = 0 \). Now the wave equation becomes

\[ \eta_t - gH V^2 \eta + f^2 \eta = f^2 \eta_0 - fH \zeta_0 \]  

(11)

Now we can proceed to solve complete initial value problems, and they have a major surprise in them. Recall the ‘homogeneous plus particular solutions’ that solve a forced o.d.e. Here they come again!

In classical fluid dynamics, there is much attention given to 2-dimensional flows with zero vorticity… ‘potential flow’. For zero viscosity, the vorticity \( \nabla_H^2 \psi \equiv \psi_{xx} = \psi_{yy} \) is conserved following the flow. This is what the physicist Richard Feynman called a ‘dry fluid’. One aspect
of these flows is that all the fluid particles are in some sense identical…none is ‘marked’ or distinguished by having special density or temperature or, in this case, vorticity. The long gravity wave equation without rotation is also describing a ‘dry’ fluid. But now with rotation, we see that fluid particles are distinct: they can be ‘marked’ or ‘recognized’ by their PV (potential vorticity). The PV is conserved following the fluid particle.

Thus consider an initial-value problem, in which the fluid near the origin (x=0, y=0) is given some initial surface displacement, \( \eta = \eta_0(x,y) \). Unless it happens that the initial conditions also have a relative vorticity \( \zeta_0 \) that cancels \( \eta_0 \) in the PV, then that initial PV will remain with the fluid. Waves can carry away energy and transmit momentum but they cannot ‘radiate’ or carry away PV, unless the fluid particles are carried away (or, viscosity can kill off the PV or some external forces might change the PV). So, we are left with a fluid circulation near the origin. This is known as geostrophic adjustment.

**Kelvin waves.**

Before going off with this new equation (11), we need to look at one more solution of the simpler form for zero initial values of \( \eta \) and \( \zeta \), that is equation (9). It is a peculiar kind of wave that doesn’t follow the normal rules of \( \exp(i k x + i l y - i \omega t) \). In this problem there are two distinct restoring forces, gravity and Coriolis. In many such problems, with two restoring forces, one finds a new class of solutions called edge waves. They arise mathematically because the boundary conditions become more complex with two restoring forces. It relates also the failure of the ‘method of images’ which is a classic way of solving wave problems with a reflecting boundary. Simply stated, the sine waves derived above are not capable of describing a general initial condition, and the Kelvin wave occurs to complete the set of waves.

Here go back to the linearized MOM equations and find the relation between velocity and surface elevation, \( \eta \), for a solution that is wavy in time (but not necessarily in x and y). With \( (u,v,\eta) \) varying like \( \exp(-i \omega t) \), the equations are

\[
\begin{align*}
-i \omega u - fv &= -g \eta_x \\
-i \omega v + fu &= -g \eta_y \\
-i \omega \eta &= -H(ux + vy)
\end{align*}
\]

These can be solved for \( u \) and \( v \) as functions of the gradients of \( \eta \). Suppose now that we have a rigid side boundary on the fluid, a vertical wall at \( y=0 \). There the boundary condition is \( v=0 \), so that the velocity vector is parallel to the wall. In terms of \( \eta \) this boundary condition is

\( -i \omega u = -g \eta_x \) \hspace{1cm} (12a)

This is peculiar because it looks like a relationship for gravity waves without Coriolis terms (with \( f=0 \)). Similarly the other MOM equation at the boundary becomes

\( f u = -g \eta_y \) \hspace{1cm} (12b)

which is the geostrophic balance equation in the y-direction. Somehow the x- and y-directions are playing very different roles here. To relieve the suspense, we find that there is a solution with \( v=0 \) not just at the boundary but everywhere in the fluid: the horizontal velocity is perfectly aligned with the wall. Under this assumption the equations are just (12a,b) everywhere in the fluid. The solution is

\[ u = U \exp(-\lambda y) \exp(ikx - i\omega t) \]
\[ \eta = \left( \frac{f \lambda}{g} \right) U \exp(\cdots) = \left( \frac{H}{g} \right)^{1/2} U \exp(\cdots) \]

where the dispersion relation is

\[ \omega^2 = gH \]

and \( \lambda^2 = gH/f^2 \). These are Kelvin waves, with the peculiar property that they are trapped near the ‘coastal’ boundary, and propagate along it with a wave speed simply equal to the non-rotating gravity wave speed. They are non-dispersive for this reason. Another remarkable feature is that **Kelvin waves propagate in only one direction, for example cyclonically around an ocean basin.**
Kelvin waves are very important in the ocean, where they express much about the ocean tides. When we take density stratification into account they become internal Kelvin waves, which are very strongly evident along the Equator. Why? There is no rigid boundary at the Equator but \( f \), the Coriolis frequency changes sign there and so the Equator becomes a virtual boundary for these waves. More correctly, a southern hemisphere Kelvin wave and a northern hemisphere twin Kelvin wave can lean on one another to make a powerful mode that propagates eastward (only) along the Equator at the speed of long gravity waves. Note that this obeys the rule above that these waves propagate cyclonically round an ocean basin. They exist in both atmosphere and ocean and are a crucial part of the El Nino/Southern Oscillation phenomenon.

**Footnote: Edge waves: the surprising result of mixed boundary conditions.** When you have some kind of waves and wish to include a boundary, you anticipate there will be reflection: waves will bounce off walls. In classic waves like light, sound or non-rotating gravity waves we find that the boundary condition for the problem is usually ‘simple’, that is, the wave function \( \eta(x,y,t) \) is zero or its normal gradient is zero at the wall. These are known as Dirichlet or Neumann boundary conditions, respectively, in applied math discussions. In these cases reflection is a simple process that can be visualized by the ‘method of images’. A candle in front of a mirror is reflected and the reflection looks like the candle. All the sine-wave components in the Fourier decomposition of the candle image reflect without a shift in phase (or if there is a shift in phase it is simple, like \( \pi \) radians). The success of the method of images tells us that the sine-waves making up the image are a ‘complete’ set, which fully describe any reasonable image, so the reflection has no surprises in it.

Consider however some boundary condition that is a mixture of the value \( \eta \) and its normal derivative \( \nabla \eta \cdot \mathbf{n} \) where \( \mathbf{n} \) is a unit vector normal to the boundary. Then there is a phase shift between a sine wave hitting the boundary and its reflected sine wave. This is exactly the case for rotating long gravity waves. It means that the method of images fails. What you see is not a perfect reflection of the ‘candle’ (that is, the incoming wave field). This failure suggests that there is a missing wave mode; something is needed to satisfy the boundary condition and complete the family of waves. This is the ‘edge wave’, of which the Kelvin wave is the most famous example. If you put a ‘paddle’ that generates waves near a rigid side boundary you get not only a reflected wave but also a Kelvin edge wave propagating away, along the boundary.

In GFD generally, these mixed boundary conditions arise frequently when there are two distinct restoring forces for the wave motion. In the case of Kelvin waves it is gravity and rotation. The Kelvin wave has a gravity-wave dispersion relation but is trapped near the boundary by Coriolis effects. With stratified fluid is buoyancy and rotation. With atmospheric waves it is compressibility of the air and buoyancy of the stratified fluid (the edge wave is the ‘Lamb wave’, a sound wave trapped by the stratification.

As an exercise it is useful to take the MOM equations for an oscillatory motion, \( u,v,\eta \) proportional to \( \exp(-i\omega t) \) and calculate the expressions for \( u \) and \( v \) in terms of \( \partial \eta / \partial x \) and \( \partial \eta / \partial y \). The result is

\[
    u = \frac{g}{f^2 - \omega^2}(i\omega \eta_x - f \eta_y), \quad v = \frac{g}{f^2 - \omega^2}(f \eta_x + i\omega \eta_y)
\]

Notice the two extreme limits: \( \omega/f >>1 \) which is simple gravity waves and \( \omega/f<<1 \) which is geotrophic balance. These show that, for example, a rigid vertical wall at \( x = 0 \) with boundary condition \( u = 0 \) at \( x = 0 \) will require a combination of \( \eta_x \) and \( \eta_y \) to vanish at \( x = 0 \). Also, these limits are a good way to check that the signs are correct.