**Geostrophic adjustment in a 1-layer fluid with a free surface: review notes**

The idealized problem solved with a $\eta = \sin(k_0x)$ initial condition can be generalized in several ways. First of all, Fourier analysis enables us to add together sine-waves of a range of wavelengths to produce new solutions from the one we have. This is the huge benefit of having linear dynamics: you can add two solutions together to get a third solution. Areas of physics governed by linear equations can have powerful theories (for example Maxwell’s equations for electromagnetic fields are basically linear until one puts them into moving, conducting fluids like the Earth’s metallic liquid core; basic theory of electrical circuits is linear, so one can built a theory based on sending individual sine-wave currents through them). So, for example Gill’s geostrophic adjustment problem with a ‘step-function’ initial condition can be found by adding together many of our sine-wave solutions. We can go farther with Fourier and add up sine-waves in all directions to make a circular initial condition, for example a Gaussian bell-curve function of radius $r$. The dispersion of this initial ‘pulse’ of fluid elevation plays out on the $x$-$t$ (or $r$-$t$) diagram, space and time. Matlab’s FFT can solve this for you in a couple of steps.

Numerical simulations like that appended to the previous lecture notes, or the m-file from Holton & Hakim’s textbook posted on the website, are useful to explore in depth.

We used the physicist Richard Feynmann’s story about ‘dry water’ (that is, water with out viscosity or vorticity that obeys Laplace’s equation) and ‘wet water’ (flows with viscosity, vorticity, shed vortices…). In our case the long hydrostatic linear wave equation was ‘dry water’ because any fluid parcel could be exchanged with any other with little effect. Add Earth’s rotation and you get ‘wet water’, where each fluid column has its own potential vorticity (PV) which is attached to it until dissipated by friction or altered by external forces. In particular, waves cannot carry away the energy in the initial field of $\eta_0$, leaving a geostrophically balanced steady circulation locked to this PV.

Now, GFD gets you only some of the way into real flows. Here, when we do experiments or look at observations it is clear that setting up the geostrophic flow is only the first step in the life-cycle of a fluid circulation. In realistic cases, the geostrophic flow will itself be unsteady over a longer time-scale. It may turn into a Rossby wave, or a pair of vortices that advect one another (as seen in the GFD lab), or chaotic geostrophic turbulence (…many such vortices interacting with one another). Or, even the generation of non-geostrophic fronts and waves through flow instability.

One of the most important results of this model is the role of the Rossby deformation radius, $L_d = c_0/f$, where $f = 2\Omega$ and $c_0$ is the horizontal propagation speed of long, hydrostatic gravity waves without rotation, hence $c_0^2 = gH_0$ ($H_0$ is the mean depth of the fluid). $L_d$ is the distance that a gravity wave propagates in one half-pendulum day, $1/f$ sec. (about 3 hrs in middle latitudes). After this short time Coriolis forces act strongly on the flow and begin to ‘arrest’ the geostrophic part and hold it in place. For the one-layer problem, the horizontal scale $L$ of $\eta$ divided by deformation radius, $(L/L_d)^2$, tells you the ratio of APE (available potential energy $\frac{1}{2} \rho g \eta^2$, to kinetic energy $\frac{1}{2} \rho H_0 (u^2 + v^2)$… note we are in the hydrostatic limit here, $(H/L)^2 \ll 1$. 

---

**GEOPHYSICAL FLUID DYNAMICS-I  OC512/AS509 2015  P.Rhines**

**LECTUREs 9-10 week 5  Stratified, rotating fluid: thermal wind**

small additions in blue
That same ratio also tells you the fraction of the initial APE that will appear as KE of the geostrophic circulation (the remainder going into long gravity-inertial waves). Large-scale initial $\eta$ fields ($L/L_d >> 1$) have great APE and are already close to geostrophic balance: little energy will leak away in waves. Small-scale initial conditions ($L/L_d << 1$) will give the initial APE and KE mostly to the transient long wave.

*Geostrophic adjustment with a flow driven by external forces.* We emphasized that ‘overturning circulation’, that is the fluid velocity components normal to the geostrophic flow, and vertical, is a key to understanding geostrophic adjustment. The problems here have come to geostrophic balance quickly, after a time $\sim f^{-1}$. However, it can take much longer to achieve balance. Imagine setting up a flow with a uniform body force $F$ acting on a one layer fluid in channel. Suppose the force acts in the x-direction and there are two rigid boundaries parallel with the x-axis. A linear friction force $-Ru, -Rv$ occurs in the momentum equations. It is a very interesting (and solvable) problem. Turning on the force at time $t=0$, the u-velocity accelerates rapidly at first, but then the velocity vector rotates to the right, and the new cross-channel velocity $v$ develops. This is like our earlier adjustment problems, however the $v$-velocity has to move fluid across the channel to the right, where it can lean on the rigid boundary. If the channel width $L >> L_d$, this is a long time indeed: the negative $v$-velocity is the overturning circulation that builds up the sloping water surface, and the Coriolis force on $v$ almost perfectly opposes the driving force $F$ for a very long time. In the sketch below the Rossby radius $L_d$ is shown, very small and hence the PE/KE $>> 1$; this alone tells us that it’s going to be a long, slow adjustment to build up the PE (by the work done by the $F u$). Note that $F$ is uniform across the channel: how would this sequence look if $F$ were concentrated at the center of the channel?
This channel problem suggests that we think about the APE/KE ratio for the observed oceans (especially) and atmospheres. Both have APE/KE $\gg 1$ (roughly 50 in oceans and 5 in the atmosphere). The energy cycle involves creation of APE by solar radiation largely (passing through thermal internal energy first). This APE is the reservoir for the KE of the actual circulation. Release of the APE is a complex process in detail but simple in principle: fluid slumps under gravity’s buoyancy forces to try to lie as horizontally as possible. In doing so it creates flow...KE. The numerical value of APE/KE suggests how big the reservoir is. If we switched off the incoming radiation, how long would circulation persist? Our own intuition is valuable: we observe the seasonal march of the atmosphere lags the sunshine by about 1 month. The ocean on the other hand could run about 10 years on its store of APE before the great lateral gyres and boundary currents slowed significantly. Such estimates require some more knowledge of energy conversion and flux and dissipation, which will come along a little later.

**Thermal wind equation.** The Taylor-Proudman theorem tells us that rotation ‘stiffens’ the fluid along lines parallel with the rotation vector. In the f-plane (flat Earth) model, this means that columns of fluid move like vertical pillars. Clouds of dye, fully 3-dimensional, begin to form 2-dimensional ribbons. Viewed from above they are thin, viewed from the side they are tall.

Adding stratification changes only the vertical MOM equation, at small Rossby number: the horizontal MOM equations remain in geostrophic balance. The vertical MOM balance is independent of this. For ‘thin layers’ of fluid, with motions having $(H/L)^2 \ll 1$ where $H$ is here the height scale of the fluid velocity and $L$ is its lateral scale, the vertical balance is hydrostatic. Our three MOM equations are as before

$$-fv = -\rho^{-1} \frac{\partial p}{\partial x}, \quad fu = -\rho^{-1} \frac{\partial p}{\partial y}, \quad 0 = - \frac{\partial p}{\partial z} - g \rho.$$
Let the fluid density have a mean vertical variation $\rho_0(z)$ plus density variations associated with flow and waves:

$$\rho = \rho_0(z) + \rho'(x, y, z, t).$$

Now the density $\rho$ is no longer a constant, yet it is taken to be constant in the horizontal MOM equation. This is a strict form of the Boussinesq approximation, valid roughly for stratification that is weak enough that the vertical length scale of variation of the density profile, $H_S \sim \rho'/(\partial \rho/\partial z)$ is much greater than the vertical length scale of variation of $u$ or $\rho'$, $H \sim u/(\partial u/\partial z)$ for example: $H/H_S \ll 1$. Compressibility is neglected here.

Take the $z$-derivative of $y$-MOM and $y$-derivative of $z$-MOM, to find

$$f\partial u/\partial z = g\rho_0^{-1} \partial \rho/\partial y,$$

and similarly with the $x$- and $z$-MOM equations give

$$f\partial v/\partial z = -g\rho_0^{-1} \partial \rho/\partial x.$$

These are known as the thermal wind equations for small Rossby number, small $1/\beta T$, and thin aspect ratio, $(H/L)^2 \ll 1$ and Boussinesq approximation $H/H_S \ll 1$. For an unstratified fluid we recover the Taylor-Proudman result $u_z=0, v_z=0$ (e.g. Vallis 2.8.4). In vector form thermal wind balance is:

$$f \frac{\partial \vec{u}}{\partial z} = \frac{\vec{g}}{\rho_0} \times \nabla \rho$$

(1)

where $\vec{g}$ is the net gravity acceleration (points downward); the right-hand screw rule is used to point to the cross-product of two vectors. Now the fluid is allowed vertical shear: the horizontal velocity can vary in the vertical direction, and with it, $w(z)$ is no longer a linear variation in $z$. Compressibility is neglected here. Thermal wind for a more general fluid is very similar, but instead of horizontal derivatives $(\partial \rho/\partial x, \partial \rho/\partial y)_{z}$ at fixed height $z$, we take horizontal derivatives along a constant pressure surface; Gill describes this in §7.7 and Vallis gives the effect of compressibility in (2.8.4), which we will introduce later.

**Thermal wind and sloping density-surfaces.** Clearly, a density surface inclined to the horizontal would ‘slump’ toward the horizontal if there were no forces to resist buoyancy. We can write (1) in terms of that slope:

$$f \frac{\partial \vec{u}}{\partial z} = \frac{\vec{g}}{\rho_0} \frac{\partial \rho_0}{\partial z} \times \nabla Z_\rho$$

$$= -N^2 \hat{z} \times \nabla Z_\rho$$

where $Z_\rho$ is the height of a constant density surface (not potential density!).

This is a graphic way to think about thermal wind balance. Thus the Gulf Stream when it flows eastward out to sea obeys

$$f u_z = N^2 \partial Z_\rho/\partial y$$

corresponding to a constant-density surface sloping up to the north…the ‘north wall’ or subpolar front.
(Don’t confuse $Z_\rho$ with the dynamic height $Z$ or geopotential $\Phi = gZ$ of a constant pressure surface).

The thermal wind equation written with pressure as a vertical coordinate is

$$f \frac{\partial u}{\partial p} = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial y} \bigg|_{p=\text{const}}$$

$$f \frac{\partial v}{\partial p} = \frac{1}{\rho^2} \frac{\partial \rho}{\partial x} \bigg|_{p=\text{const}}$$
The two figures above show the temperature in an east-west section by L.V. Worthington, from eastern Canada to Greenland to Ireland (the ship track is in red on the image of Greenland, the subpolar Atlantic and Europe. Constant density surfaces are not shown, but have similar appearance and slopes.
The figure above shows a section (marked yellow in the map above) crossing the oceanic polar front at the Iceland-Faroe Ridge, between Greenland and Norway. The density contours are in black, with cold, fresh polar waters at the right (north) and warm, saline water from the low latitude Atlantic at the left (south). The colors show thermal wind velocity, red out of the figure, blue into the figure. From Beaird, Rhines & Eriksen, Seaglider observations, 2009.

Figure above shows salinity at the polar front (Seaglider track in inset). The warm temperature of the water south of the front makes it less dense than the cold water to the north, despite the opposing salinity difference.

A fluid built as a ‘parfait’ (a pretty multi-colored drink) of many layers can be a very good model of a stratified fluid. Each layer The deep cold blue water is flowing generally southward, winding around Iceland, Greenland and Labrador, while the warm (red) waters from the subtropics flow northward. The currents in both directions have thermal-wind shear (test this: the deep currents increase in strength downward while the upper level currents tend to increase in strength upward). They ‘lean on’ the sloping topography to balance the Coriolis force.

To get a feel for the ‘numbers’, consider thermal balance for an ideal gas. With $p = \rho RT$, 

$$\frac{\partial u}{\partial z} = \frac{g}{f} \frac{\partial \rho}{\partial y} = \frac{\partial T}{\partial y} = 327 \frac{\partial T}{\partial y};$$

for the choice $T=300K$, $f=10^{-4} \text{sec}^{-1}$.

(The thermal expansion coefficient for an ideal gas, define as $\alpha = -\rho^{-1} \frac{\partial \rho}{\partial T}|_{p=\text{const}}$ is simply $1/T$.) A jet stream with northward temperature gradient of -10K per 100km has a zonal velocity increasing by 32.7 m sec^{-1} per km of height.

For the ocean, $\alpha = 2.97 \times 10^{-4}$ at $T=25C$, salinity=35 psu. In this case

$$\frac{\partial u}{\partial z} = -\frac{g}{f} \frac{\partial T}{\partial y} \approx (9.8*2.97*10^{-4}/10^{-4}) \frac{\partial T}{\partial y} = 29 \frac{\partial T}{\partial y}$$

The Gulf Stream, with an 10C temperature drop in 100 km, has its u-velocity increase by 0.29 m sec^{-1} per 100m of vertical height change. Actually in the ocean it may be more convenient to use
the thermal wind expressed in terms of the slope of the constant-density surfaces, and the buoyancy frequency, \( N \): 
\[
\frac{\partial u}{\partial z} = \left( \frac{N^2}{f} \right) \frac{\partial Z}{\partial y}
\]

**Layered model of thermal wind equation.** A fluid built as a ‘parfait’ (a pretty multi-colored drink) of many layers can be a very good model of a stratified fluid. Each layer behaves as in the one-layer model above, yet the hydrostatic relation couples the layers together. This is described in Gill §6.2, Vallis §3.4. Consider a fluid with an upper layer of density \( \rho_1 \), lying above a lower layer of mean thickness \( H_2 \) and density \( \rho_2 \). The upper surface is free, at vertical position \( z = \eta_S(x,y,t) \) and the interface between layers is at \( z = -H_1 + \eta_I(x,y,t) \). As usual we consider motions with ‘thin’ aspect ratio, \( (H_1/L)^2 \ll 1 \), which puts the pressure field in hydrostatic balance (pressure = weight of fluid overhead per m\(^2\) horizontal area). As earlier, the pressure in the upper layer is

\[
p_1 = \rho_1 g (\eta_S - z)
\]

The pressure in the lower layer is found by adding up this column’s weight:

\[
p_2 = p_1 + g (\rho_2 - \rho_1) (\eta_1 - (z + H_1))
\]

The horizontal pressure force is thus

\[
\frac{\partial p_1}{\partial x} = \rho_1 g \frac{\partial \eta_S}{\partial x}
\]

\[
\frac{\partial p_2}{\partial x} = \frac{\partial p_1}{\partial x} + g (\rho_2 - \rho_1) \frac{\partial \eta_I}{\partial x}
\]

and similarly for the \( y \)-direction. The MOM conservation equations for steady, geostrophic balance become

\[
-f v_1 = -\frac{\partial p_1}{\rho_1 \partial x} = -g \frac{\partial \eta_S}{\partial x}
\]

\[
-f v_2 = -\frac{\partial p_2}{\rho_2 \partial x} = -g \frac{\rho_1}{\rho_2} \frac{\partial \eta_S}{\partial x} - g' \frac{\partial \eta_I}{\partial x}
\]

where \( g' = g(\rho_2 - \rho_1)/\rho_2 \).

Suppose that \( \Delta \rho/\rho_2 \ll 1 \). Then, taking the difference between the upper-layer and lower layer MOM balance,

\[
f (v_1 - v_2) = -g' \frac{\partial \eta_I}{\partial x}
\]

\[
f (u_1 - u_2) = +g' \frac{\partial \eta_I}{\partial y}
\]

This is Margules relation, the version for a layered density of thermal wind balance. It is often taken as a model set up for a density front.

Notice that the vector velocity difference between the two layers points along height-contours of the interface field. In just the same way, thermal wind balance equation shows that the difference in horizontal velocity between two nearby heights \( z \) and \( z + \delta z \) points along height contours of a constant-density surface.

**Baroclinic and barotropic modes.** The interface elevation \( \eta_I \) determines the geostrophic velocity difference between adjacent layers. If \( \eta_I = 0 \), \( u_1 = u_2 \) and the flow is called barotropic. The only PE associated with the barotropic mode comes from the free-surface elevation. On the other hand the vertical shear \( u_1 - u_2 \) is entirely associated with the interface slope, and this is the
baroclinic mode. When \( \Delta \rho / \rho \ll 1 \) the baroclinic mode is dominated by the interface elevation (the surface ‘expression’ \( \eta_S \) is small in its dynamic impact). The baroclinic mode is characterized by zero-vertically averaged horizontal mass flux: here, in a 2-layer model \( H_1 u_1 = -H_2 u_2 \).

These distinctions become useful: consider a cyclonic eddy. Will it have a cold core or a warm core? There are many aspects to this question but foremost is the mix of baroclinic and barotropic modes making up the vertical structure of the eddy. An eddy cold at the center will have density surfaces that slope upward toward its center. Thus, in geostrophic + hydrostatic balance, its azimuthal velocity will become more cyclonic with height. If however we add a barotropic mode with anticyclonic rotation (that is an eddy with no thermal wind, no vertical shear of horizontal velocity) it can reverse the sense of rotation and we have a cold-core anticyclone (in which the anticyclonic azimuthal flow weakens with increasing height, perhaps even reversing to make an upper-level cyclone). Clearly we need dynamical ideas that tell us how these different vertical modes are generated in the heat-engine of the atmosphere/ocean system.

But it is clear from thermal-wind balance that if the barotropic velocity is weak, a cyclonic eddy with strongest velocity at the top of the fluid will have a cold core. Its isopycnal surfaces dome upward (in the ocean, low-salinity water can contribute as well as cold temperature). Conversely, a cyclone at the base of the fluid will have a warm core if the barotropic mode is weak. The crucial overturning circulations that decide this would, for example, be a horizontal convergence (i.e., inflow) caused by heating the low-level fluid, which must result in cyclonic vorticity near the base. This discussion needs much further development however!

Wave theory for oceans or atmosphere involves related issues of barotropic and baroclinic velocity. Horizontally propagating Rossby waves, internal (gravity/inertial) waves, and key unstable wave modes, have both barotropic and baroclinic structure.

1 ½ Layer model of a stratified fluid: density compensation. A simple consequence of the layered density model is that the surface elevation and interface elevation may oppose one another, cancelling out the lower-layer pressure gradient. This would mean that the lower layer would have no geostrophic flow at all! It sounds unlikely but in fact when the lower layer is much thicker than the upper layer, \( H_2 >> H_1 \), then the baroclinic mode velocity is concentrated in the upper layer \( (u_1/u_2 = H_2/H_1 ) \). When this is the case

\[
\eta_S = -\frac{\Delta \rho}{\rho} \eta_I,
\]

and the density interface is a ‘mirror image’ of the free surface elevation, yet vastly greater in amplitude. In the GFD lab this is easy to simulate: the thin upper layer acts like a 1-layer model with a free surface yet with a greatly reduced effective gravity, \( g' = g \Delta \rho / \rho \). Thus we can look at a buoyant blob of fresh water floating on top of a salty, deep lower layer and spinning up to geostrophic vortex after sending off inertial-gravity waves with speeds of \( \sqrt{g' H_1} \) or less… just a few cm sec\(^{-1}\).

Ironically, the Rossby waves that dominate the eddy motions of the upper few hundred m. of the world ocean are close to these ‘compensated’ 1 ½ layer modes, yet they are seen by their
surface elevation \( \eta_S \) by orbiting satellite radar altimeters, even though \( \eta_S \) is dynamically small. A hydrographic section across the Gulf Stream shows the steep upslope of isopycnal (constant-density) surfaces toward the North; this is consistent with the eastward velocity increasing upward but surprises us, since the denser water to the north would seem to provide a pressure gradient balancing a westward flowing current (its Coriolis force would be northward). But this internal density slope is **compensated** by a tilted sea surface, rising about 1m. toward the south. Thus the thermal wind shear is there to **reduce** the strength of the Gulf Stream with depth, \( \partial u / \partial z > 0 \). This scenario comes out naturally from a geostrophic adjustment calculation.

**The buoyancy frequency, N.** This topic is very close to describing the real world of atmosphere/ocean circulation and internal waves, in the case of hydrostatic vertical MOM balance, \((H/L)^2 << 1\). To introduce it we must describe the buoyancy effect of a stratification carefully. With a mean stratification \( \rho_0(z) \), imagine moving a fluid parcel upward through a distance \( \delta z \). If this parcel exchanges no heat with its surroundings (or salt, in the ocean), it will find its own density differs from the ambient surrounding fluid by an amount

\[
-d\rho_0 \delta z + \frac{\partial \rho_0}{\partial \rho} \frac{d\rho_0}{dz} \delta z
\]

where \( \rho_0 \) is the hydrostatic pressure \(-g \rho_0 \delta z \). The signs are tricky (\( \rho \) and \( \rho \) decrease with increasing \( z \)). The second term accounts for compressibility of the fluid parcel: as it rises to a level of lower pressure, it expands. Compressibility for adiabatic motions also arises in sound waves, and we use the speed of sound, \( c_s^2 = (\partial p / \partial \rho)_{\text{entropy}=\text{const}} \) to rewrite this as

\[
-\frac{d\rho_0}{dz} \delta z + c_s^{-2} \frac{d\rho_0}{dz} \delta z = -\frac{d\rho_0}{dz} \delta z - g \rho_0 c_s^{-2} \delta z
\]

Multiply this density difference by \( g \) to give the buoyancy force on the parcel. This force restores the parcel to its original depth. An oscillator equation can be written, with \( g \) times this density difference as the vertical force and \( (c_s^2 / \partial t^2)(\delta z) \) as the acceleration, \( \partial w / \partial t \). The result is an oscillation:

\[
(c_s^2 / \partial t^2)(\delta z) + N^2 \delta z = 0
\]

\[
N^2 = \left( -\frac{g}{\rho_0} \frac{d\rho_0}{dz} - \frac{g^2}{c_s^2} \right)
\]

The signs can be confusing: \( dp_0 / dz \) is negative so the first term is positive. The second term **diminishes** the restoring force, hence reducing the frequency.

The buoyancy frequency \( N \) and Coriolis frequency \( f \) express the restoring forces due to stratification and rotation. The period \( 2\pi / N \) of oscillation varies from 20 min or so to 3 hrs in the oceans, and from 5 to 20 minutes in the atmosphere. The higher frequency of buoyancy oscillations in the atmosphere, compared with oceans, is largely responsible for the much greater horizontal scale of the dominant eddies in the atmosphere, as compared with oceans. The ratio \( N/f \) usually exceeds unity and internal waves fall in between the two frequencies.

The ratio of the second term above, divided by the first term, is
\( \frac{gH_s}{c^2} \approx 1 \)

for air with a 10 km scale height, so compressibility cannot be neglected except for idealized, simplified models. For the oceans this ratio is between 1 and 2 as well.

**Buoyancy frequency and potential density.** The temperature, following a fluid parcel, changes due to adiabatic compression or expansion. If we imagine bringing fluid to a particular reference height, \( z_{\text{ref}} \), the potential temperature \( \theta \) is defined as the temperature that the parcel would have if moved adiabatically to \( z_{\text{ref}} \). To relate \( \theta \) and \( T \), use the 1st Law of thermodynamics (the internal energy equation) written with entropy, \( \eta \), as one of the two variables:

\[
C_p \delta T = T \delta \eta + \frac{1}{\rho} \delta p
\]

If \( \delta \eta = 0 \) then

\[
C_p \delta T / T = \frac{R}{\rho} \delta p
\]

using \( p = \rho RT \). Integrate this equation from pressure \( p \), temperature \( T \) to the reference pressure \( p_r \) and the temperature at that level, which is \( \theta \):

\[
\theta = T \left( \frac{P_r}{P} \right)^{R/C_p} = T \left( \frac{P_r}{P} \right)^{(\gamma-1)/\gamma}
\]

where \( \gamma = 1.4 \) for a perfect dry diatomic gas (like air).

Now we can express the buoyancy frequency, \( N \), in terms of \( \theta \). The parcel argument shows that, with the effects of adiabatic expansion already incorporated in \( \theta \),

\[
\delta p = \rho \alpha (T_{\text{parcel}} - T_{\text{surroundings}})
\]

\[
= \rho \alpha \frac{d\theta}{dz} \delta z
\]

where \( \alpha = -(1/\rho) \partial \rho / \partial T |_{\text{p=const}} \). For a perfect gas, \( \alpha = 1/T \). The buoyancy force on the parcel is \( -g \rho \alpha \delta T \), and hence the buoyancy frequency, squared, is this

\[
N^2 = \left( \frac{g}{\rho} \right) \frac{\delta p}{\delta z}
\]

\[
= g \alpha \frac{d\theta}{dz}
\]

\[
= \frac{g}{\theta} \frac{d\theta}{dz}
\]

In ocean dynamics, potential temperature is defined based on the measured equation of state, which has nonlinear dependence on temperature, \( T \).

**Potential density** is similarly defined as the density the fluid would have if removed to a particular reference pressure (‘altitude’), \( p_r \). Thus it is a function of position, time and reference pressure

\[
\rho_p(x,y,z,t,p_r)
\]
In oceanography, the choice of reference pressure is important, owing to the nonlinear dependence of density on (T,S,p). The parcel argument above carried out for potential density gives the simple result,

\[ N^2 = -\frac{g}{\rho} \frac{d\rho}{dz} \]

**Thermal wind balance and vorticity.** In essence, the thermal wind equation is about vorticity: the *horizontal* vorticity of a fluid produced by the ‘twisting’ by pressure and buoyancy forces; see Gill §7.11. The vorticity equation is found by taking the curl of the vector momentum equation (or cross-differentiating the three scalar momentum equations). The new term due to density differences is on the right-hand side:

\[
\frac{D\omega}{Dt} = ((\omega + 2\Omega) \cdot \nabla)\mu - \nabla \times \left( \frac{\nabla p}{\rho} \right)
\]

Note that the curl of a gradient vanishes identically, the new term comes from \( \nabla p \times \nabla (1/\rho) \). The fundamental theorem of vorticity conservation (that vortex tubes conserve their ‘strength’ or vorticity times cross-section area) is no longer true. Wherever isobars and isopycnals (constant pressure and density surfaces) intersect, twisting forces produce vorticity. It seems strange that the pressure gradient can do this, but consider a sphere of stratified fluid: the pressure force on the surface of the sphere acts through its geometric center, yet the center of mass is some distance away from that point due to the stratification. Thus there is a torque of the pressure force about the center of mass, which will cause the sphere to rotate. More simply put, the dense fluid will tend to fall to the lowest possible position, and this involves rotating the sphere.

**Buoyancy/pressure forces produce mostly horizontal vorticity.** Notice that both \( \nabla p \) and \( \nabla \rho \) are nearly vertical, in the large-scale motions we are discussing: the dominant pressure field is in hydrostatic balance. Therefore their cross-product is nearly horizontal,

\[ \nabla \rho \times \nabla p/\rho^2 \approx (\zeta \rho_y / \rho_0, \zeta \rho_x / \rho_0, 0) \]

With the Boussinesq approximation, buoyancy twisting makes horizontal vorticity. This is a fundamental property, for it allows the vertical vorticity equation to be free of these buoyancy effects. We observe vertical shears far in excess of \( f \), and yet the vertical vorticity of synoptic-scale (ocean mesoscale) eddies is usually far less than \( f (\zeta / f \sim \text{Ro}, \text{the Rossby number}) \). Indeed, smaller scale nonlinear eddies with \( \text{Ro} \geq 1 \) are also common, yet they often evolve back toward balance through continual geostrophic adjustment. There must be a flow of energy in oceans and atmosphere from ‘balanced’ flow to unbalanced internal waves and 3D turbulence, in order to do the eventual dissipation. Yet the whole sense of geostrophic flow is that energy dissipation by turbulence is greatly reduced by geostrophic effects, since vortex stretching is resisted by Coriolis effects: the ‘stiffness’ of rotating fluids.

Writing the three components of vorticity as \((\omega_x, \omega_y, \zeta)\), (with \( \text{Ro} \ll 1 \); \( (H/L)^2 \ll 1 \), \( H/H_s \ll 1 \)), the vorticity equation becomes
\[
\begin{align*}
\frac{D\zeta}{Dt} &= fu\zeta; \\
\frac{D\omega}{Dt}^x &= fu - (g / \rho_0)\rho'^y; \quad \omega_x \equiv w - v \\
\frac{D\omega}{Dt}^y &= fu + (g / \rho_0)\rho'^x; \quad \omega_y \equiv u - w
\end{align*}
\]

Thermal wind balance now appears embedded in the horizontal vorticity equation, and dominates when the $D/Dt$ term is small. Notice that with geostrophic/hydrostatic scaling, the vertical vorticity equation is unchanged by density stratification: this means that the 'stiffness' of a rotating fluid is still present, resisting stretching of the vertical vorticity, $\zeta$.

If we don’t make the approximation just above, the initial vorticity equation above, the new twisting term has two parts, for example the x-component in the vorticity equation is $p_y \rho - p_z \rho_y$, a combination known as the Jacobian of $p$ and $\rho$ with respect to $y$ and $z$, $J_{yz}(p, \rho)$. It happens that this expression leads to a more exact thermal-wind balance where the density gradient is calculated along constant-pressure surfaces rather than horizontal ($z=$constant) surfaces:

\[
f \frac{\partial \vec{u}}{\partial \zeta} = \frac{\partial}{\partial \zeta} \rho \times \nabla \rho \bigg|_{p=const}
\]

Because constant pressure surfaces are more nearly horizontal than constant density surfaces for geostrophic/hydrostatic flows, this correction is small. Gill’s section on isobaric coordinates §6.17 discusses similar issues.

At small scale, non-geostrophic flows can and do break these rules. Flows over small hills, and tornadoes, both seem to produce vertical vorticity by tipping the nearly horizontal vortex lines of the boundary layer, so that they have a vertical component (a ‘hair-pin’ shape). This is readily observed in the lab. See research papers of Christoph Schär, Richard Rotunno, Dale Durran. Similar processes must occur in the oceanic bottom boundary layer and also in the boundary layer at the surface...see papers of Leif Thomas.

It is tempting to make the analogy to gyroscopes at this point: a gyroscope with an axis tilted away from the vertical experiences a gravity torque, which makes it precess about the vertical (rather than fall over!). However that isn’t the same as thermal wind balance despite the superficial resemblance. A shear flow $\partial u / \partial z$ is trying to tip over the huge planetary vorticity (let’s say that is vertical) to make horizontal vorticity $\omega_x$. But, instead, buoyancy twisting $g\rho' y$ cancels out that new $x$-component of vorticity.

This sets the stage for the final derivations of GFD-1: stratified dynamics of geostrophic circulation, internal waves and quasi-geostrophic Rossby waves. Once geostrophic adjustment has occurred, the flows we have looked at become steady, because of the 1-dimensional variation of $u,v,\eta$. Realistic 2D and 3D flows go through this same adjustment to balance, but that is just the beginning of their life cycle. Rossby waves, instability of large-scale flow, interaction of
vortices are the ‘weather’ and ‘climate’ we are interested in, and their time-space development is the next major subject of GFD.