We will work on some of these problems in class. Turn in written solutions/discussion for 5 of the 8 problems below. You may discuss these with colleagues but solve as much as possible on your own.

1. Show that a column of dry atmospheric air has internal energy $E$ and gravitational potential energy ($P_e$) with ratio $E / P_e = 1/(\gamma - 1)$ where $\gamma \equiv C_p/C_v$.

   For diatomic molecules dominating air, $\gamma = 7/5 = 1.4$, so $P_e/E = 0.4$ as in the similar example worked out in class, which looked at small changes $\delta E/\delta (P_e)$ when the air column was heated by a given amount $\delta^*Q$. $C_p$ and $C_v$ are specific heat capacities at constant pressure and volume, respectively. $C_p = 1000 \text{ J kg}^{-1} \text{ K}^{-1}$

   In class we showed this for small changes in ideal gas heated in a box, with and without expansion allowed by a moveable lid (the weight of the lid being lifted was part of the potential energy $P_e$ whereas here the atmosphere simply expands upward when heated...with no lid).

   $E = \int \rho C_v T \, dz$ ; $P_e = \int \rho g z \, dz$ with the integrals extending over the whole height of the atmosphere from $z=0$ up to where $p$ goes to $0$.

   The equation of state $p = \rho RT$ and hydrostatic balance, $dp/dz = -\rho g$ can be used to relate the two integrals. Here we use $z=0$ as the arbitrary reference for $P_e$, and temperature = zero K is the reference for $T$, where $p$ goes to zero.

2. We argued that the result of problem 1 is that for the climate system the atmosphere acts like the expandable working gas of a heat engine, while the ocean acts as a large ‘reservoir’ of heat.

   Find the ratio of small changes in internal energy and potential energy, $\delta E / \delta (P_e)$, when for a layer at the top of the ocean heated by the sun and atmosphere. Treat the layer as a slab of uniform temperature $T$ and thickness $h$. Then,$$
   \delta E = \rho h C_p \delta T \quad \text{and} \quad \delta P_e = \int_0^h \rho g z \, dz = \int_0^h (\rho \int_0^h g z \, dz)
$$

   where $\rho$ is uniform in the layer. Note that as it is heated $h$ increases but the mass $\rho h$ per unit area is constant.

   Here instead of an ideal gas we have $C_p = 4000 \text{ J kg}^{-1} \text{ K}^{-1}$ and thermal expansion coefficient measured empirically: $\alpha = -\rho^{-1} \partial \rho / \partial T = 2 \times 10^{-4}$ (approximately, at temperature 20°C). In calculations like this it’s good to check that the units are the same in the two expressions for energy change (energy per unit surface area has dimensions mass x velocity²/length² (= kg sec⁻²) so $C_v$ and $C_p$ have dimensions sec⁻² K⁻¹ where “K” means degrees Kelvin.

3. Compare the Carnot heat engine with a heat engine whose cycle has two isobaric (constant-pressure) and two adiabatic curves on the P-V (pressure-volume) plane. (The Carnot cycle instead is made up of two isothermal and two adiabatic curves.)

   We are heating (and expanding) at constant pressure, expanding adiabatically, cooling (and contracting) at constant pressure, then contracting adiabatically back to where we started. What is its efficiency (ratio of work done to heat input)? Draw the cycle on the pressure-volume diagram and temperature-entropy diagram.
The motivation is to show explicitly the mechanical energy gain from the heat engine. This is a box of air with a moveable lid. A weight is placed on the lid, the air is heated and expands upward at constant pressure. Then the weight is slid onto a shelf, and the gas expands further without heating (adiabatically, at constant entropy). The gas is then cooled under constant pressure and the lid drops back down. Another weight is slid onto the lid making it drop further, adiabatically. The lifting of weights is the mechanical energy output.

4. We remarked that it’s hard to warm your coffee by stirring it. Make a calculation demonstrating this, crudely estimating the temperature change caused by dissipating the kinetic energy of swirling coffee. The conversion from mechanical to thermal energy then leads to an warming proportional to the specific heat capacity at constant pressure, \( C_p = 4000 \text{ J kg}^{-1} \text{K}^{-1} \).

5. Why are steam engines useful? Consider the toy steam boat in the GFD lab. A small metal chamber about 1 cm x 1 cm x 0.5 cm in volume was connected to two pipes that extended like twin exhaust pipes out the back, underwater. With water filling the chamber, a candle heated it until the water boiled. The big expansion of volume blew the liquid water in one of the pipes out the back, which sucked new cold water into the other pipe, refilling the chamber...and so on.

If the chamber and pipes were above the water surface, so that this became a dry-air heat engine, not involving water, would it have worked? Make some rough quantitative estimates. The heat absorbed by evaporating water (latent heat coefficient) is \( 2.25 \times 10^6 \text{ J kg}^{-1} \) for water near boiling.

6. A pot with 1 kg of water on a stove is heated from 20°C to 100°C with a steady 1 kilowatt heating. How long does it take? Then it boils until the pot is dry. How long does that take?

7. The generation of mechanical energy by a heat engine is at most \( (T_w - T_c)/T_w \) which is the efficiency of a Carnot cycle heat engine. \( T_w \) and \( T_c \) are the temperatures of the warm heat input and cold heat output reservoirs, respectively. Suppose the atmosphere has a temperature difference 30°C between tropics, where heat is injected and polar regions where heat is radiated to space (of course heat is radiated to space at all latitudes, but the heat engine works off the variation from the mean ). Suppose also the heat flow through the system from tropics to polar regions is \( 10 \times 10^{15} \text{ Watts} \) (that is, 10 petaWatts).

- What would be the rate of production of kinetic + potential energy of the atmosphere/ocean system if it had the efficiency of a Carnot cycle heat engine?
- The ratio of \( Pe/Ke \) (potential energy/kinetic energy) is about 5 for the atmosphere. How much energy would there be in the winds (and what would be their mean speed) if the dissipation time for kinetic energy is 5 days? The mass of the atmosphere is \( 5.3 \times 10^{18} \text{ kg} \).

8. Thank you, oceans. Levitus 2012 estimates that 93% of the extra heating of the Earth during the past 50 yrs of global warming has ended up in the oceans. The amount is estimated to be \( 2.5 \times 10^{23} \text{ Joules} \). If this extra thermal energy instead stayed in the atmosphere, how much would that warm the atmosphere on average? Of course the atmosphere would radiate a lot of this excess heating away to space, but first assume it does not.

IPCC estimates of greenhouse gas induced global warming, accounting for all the feedbacks in the system, are of order 1 Watt m\(^{-2}\) extra heat gain, averaged over the Earth’s surface. This compares with the average solar energy flux of \( \sim 1368 \text{ Watts m}^{-2} \) in empty space above the Earth with \( 1/4 \) that being the average incoming heat flux at the top of the atmosphere (432 Watts m\(^{-2}\)). An estimated 168 Watts m\(^{-2}\) of solar radiation is absorbed at the surface (70% of which is ocean). With the tropics receiving much more than polar regions, the atmosphere/ocean circulation transports poleward thermal energy flux peaking at roughly \( 10 \times 10^{15} \text{ Watts} \) (that is, 10 petaWatts) at 30\(^0\) latitude (\( \sim 5 \text{ pW} \) in each of the two hemispheres).