GFD 1 Winter Q. 2015  Problem Set 2  out: Friday 23 Jan
back: Friday 30 Jan

Turn in any 4 of the 8 problems, use the rest for study. The last two relate to Gill Ch. 7 and will be discussed on Tues Jan 27; others will also be discussed in class. Problem 2.2, while long to describe is actually a short calculation.

2.1 Suppose the Hadley circulation of the atmosphere to be a symmetrical overturning (uniform east/west around the globe). If it rises due to heating from cumulus convection near the Equator, flowing poleward to 20° latitude, then sinking and returning equatorward. The upper flow poleward approximately conserves angular momentum relative to the Earth axis. The lower branch is retarded by surface friction.

o What is the zonal wind velocity as a function of latitude φ or north–south distant y in the upper poleward flowing branch, assuming it to be zero at the Equator?

o Even though the vertical component of \( 2 \Omega = f = 2\Omega \sin \phi \) vanishes at the Equator, it has an effect in tropical latitudes, where it varies almost linearly with latitude: \( f \approx \beta y \) where \( \beta = df/dy \) at the Equator. Qualitatively, how will a free particle skating on a rotating sphere behave if given an initial velocity near the Equator, subject only to a Coriolis force with east- and north-components \( \beta y (v_x, -u) \)? This is analogous to the slab of fluid driven by a constant force with constant \( f \) worked out in class.

2.2 The length of the Earth day is observed to have an annual cycle of close to 1 millisecond, peak to peak (figure below). Considering the effect of the atmospheric circulation only, why do you think this happens? Assume that the angular momentum of the atmosphere + solid Earth is conserved through the year.

o Write the integral expression for the atmosphere’s angular momentum about the rotation axis assuming a zonal wind distribution \( u(\phi, z) \) and air density \( \rho(\phi, z) \) where \( \phi \) is latitude.

Now we make some crude assumptions: suppose the atmosphere rotates with one angular velocity, like a solid body. Make a rough estimate of the seasonal cycle of zonal winds by conserving angular momentum \( L \) of solid Earth + atmosphere,

\[
L_E + L_A = I_E\Omega + I_A (\Omega + \omega) \tag{1}
\]

where \( \Omega \) is the Earth’s angular velocity and \( \omega \) is the angular velocity of atmosphere, observed in the rotating frame of reference. “\( I \)” is explained as follows:

A single mass has angular momentum \( m r u = m r^2 \omega \) where \( \omega \) is its angular velocity, \( u/r \). A rotating, symmetrical solid body has angular momentum that is the integral of this expression over all the elements of mass \( dm \): \( \iint r^2 dm \) where \( r_i \) is the distance of each mass element to the Earth axis. \textit{Here we take \( \omega \) to be uniform, the same for all elements of mass, ignoring all the variation in zonal winds.}

We call \( I = \iint r^2 dm \) the moment of inertia of a rigid body and for a spherical solid Earth of uniform density it is simply

\[
I_E = 2/5 M_E a^2,
\]

\( a \) being the Earth radius and \( M_E \) its mass. For the atmosphere, a thin spherical shell, the moment of inertia is \( I_A = 2/3 M_A a^2 \).

o Use expression (1) to estimate the small change \( \delta \omega \) in the atmospheric angular velocity and hence zonal wind velocity \( u = r_i \omega \) due to a small change in angular velocity of the solid Earth, \( \delta \Omega \) corresponding to 1 millisecond change in the length of the day. Again, we are crudely assuming the atmosphere to be a thin spherical shell of uniform density rotating with uniform angular velocity \( \omega \) relative to the solid Earth.

o Given the actual variation of winds and air density \( \rho \) with latitude and altitude, where do you think the most important wind changes occur, causing this remarkably precisely known annual cycle of \( \Omega \).
The mass of the atmosphere is \( 5.3 \times 10^{18} \) kg, the mass \( M \) of the solid Earth is \( 6.0 \times 10^{24} \) kg, and Earth mean radius, \( a \), is 6371 km (Gill Appendix 2).

Left figure shows the fluctuation of the length of the day from Gross 2002, Geodesy in the New Millennium, Springer. The longer period variation in \( \Omega \) is thought to be due to interaction of the liquid iron core of the Earth with its mantle. The righthand figure shows the variation in the length of the day from astronomical measurements (solid) and inferred from atmospheric angular momentum change over 9 months (Hide 1980).

2.3 Poleward moisture flux. Evaporation at the sea surface removes about 1 m per year as a global average, which rains out both locally and far away; the evaporation is greatest in the tropics and subtropics. Using the latent heat values for water (see Gill §3.4, eqn 3.4.6),

- how much power in Watts is absorbed to evaporate this much water? Use a sea surface temperature of 25°C. The poleward heat flux (carried by the MOCs of A and O together) is about 4 petaWatts \( (4 \times 10^{15} \text{ W}) \) at its maximum; how does this compare with the latent heat flux from the sea surface?
- What is the mass transport of water vapor in Sverdrups, corresponding to 2 pW of poleward flux of latent heat? If the total water content in the air is 2.5 cm of liquid water, what would be the average mass transport poleward of air + water vapor in Sverdrups?
- Comment on the relationship between these two numbers (the evaporative energy per sec versus the poleward flux of heat).

Gill gives in Appendix 2 some more useful numbers. A number not given there is the number of seconds in a year: \( \pi \times 10^7 \) very closely! The latent heat of evaporation of seawater is about \( 2.5 \times 10^6 \text{ J kg}^{-1} \). 1 Watt = 1 J sec\(^{-1} \). 1 Sverdup of mass transport = 1 megatonne sec\(^{-1} \) = \( 10^9 \) kg sec\(^{-1} \).

2.4 The ocean and atmosphere both have meridional overturning circulations (MOCs). The atmosphere is more energetic, however the ocean can lean on the solid Earth to balance the Coriolis force on its north-south velocity. This seemingly should make it efficient as transporting heat.

- The MOC of the North Atlantic Ocean transports about 18 Sverdrups of mass. If the potential temperature upper ocean (moving poleward) averages 20°C and \( \Theta \) of the deep ocean (moving southward) averages 2°C, estimate the poleward heat flux in Watts (or better, petaWatts, \( 1 \text{ pW} = 10^{15} \text{ W} \)). How does this compare with the observations that peak at about 1.5 pW at about 20°N latitude?
- With reference to Gill’s §4.8, how is it that the atmospheric MOC involves cold upper tropospheric air moving poleward and warmer lower atmospheric air moving equatorward: how does this achieve poleward heat flux (of about 4 pW)?

2.5 Forcing a slab of fluid with Coriolis effects and friction.

Consider the inertial oscillation plus mean flow problem done in class (and in the lecture notes). Add a friction force \((-Ru, -Ru)\) so that the MOM equations are

\[
\begin{align*}
\frac{du}{dt} - f v &= -Ru \\
\frac{dv}{dt} + fu &= F_o - Rv \\
w &= 0
\end{align*}
\]

where \( R \) is a constant coefficient of friction and \( F_o \) is the constant force driving the flow (maybe a wind stress if this is an ocean).

- Solve for the steady part of the solution for \( u,v \) velocities (the ‘particular’ solution of this inhomogeneous equation set).
- Then solve the homogeneous equations to give a complete solution for the same initial conditions: \( u=0, v=0 @ \text{t}=0 \). Sketch the solution and discuss. In solving for the ‘wave’ part you may want to add the extra boundary condition that \( u \) and \( v \) are finite (not infinite) as \( t = \rightarrow \infty \).

Hint: a nice mathematical idea, when you see a combination like \((\partial / \partial t) + R)u\) is to notice that if you multiply \( 1 \) by \( e^{Rt} \) you can write \( e^{Rt}(u + Ru) = \partial / \partial t (e^{Rt}u) \) and similarly with eqn. 2. Thus changing variables to \( u^* = e^{Rt}u \) etc. can simplify the homogeneous equations (and solving the steady, particular equation is easy as well).

2.6 Making vortices in a rotating fluid.

In the lab we injected or sucked out (‘dejected’?) fluid with a rubber bulb, to make eddies. Suppose you introduce fluid \textit{steadily} from a porous cylindrical boundary at radius \( r = r_o \), uniformly with depth, and it spreads out 2-dimensionally, symmetrically with radial velocity...
u(r) and azimuthal (round and round) velocity v(r). It spins up a vortex due to the presence of background rotation $\Omega$.

- calculate the azimuthal (round-and-round) velocity as a function of r, assuming the fluid conserves absolute angular momentum $r(v' + \Omega r)$ where $v'$ is the azimuthal velocity in the rotating reference frame. The fluid enters from the boundary at $r_0$ with $v'=0$. Is this flow nearly geostrophic?

- when the flow was started it pushed fluid outward, conserving volume flux (per unit depth) $2\pi ru$, what is velocity $v'(r,t)$ of this 'old' fluid for the time before the 'new' fluid arrives from the source?

2.7 Development of geostrophic flow from initially unbalanced conditions.

As in Gill's problem in Ch. 7 which started with a step-function initial condition for the surface elevation $\eta(x,t)$ of the fluid, start instead with a sinusoidal variation of $\eta(x, t=0)$:

$$\eta = A \sin k_0 x , u = 0, v = 0 \quad @ \quad t=0$$

- Solve for the steady part of the solution ('particular' solution) and then the time-dependent ('homogeneous' or wavy part). Sketch the surface $\eta$ profile and the velocity field. The steady part is in geostrophic balance. The total depth is $H + \eta$.

- Calculate the kinetic and potential energies $KE$ and $PE$ for the steady part of the solution and compare these with the $PE$ of the initial condition.

- Discuss the dependence of the ratio $PE/KE$ on the length scale $L$ of the initial free surface profile ($L \sim 1/k_0$).

2.8 Another geostrophic adjustment problem for a finite width initial pattern of surface elevation.

- Solve for just the steady, geostrophically balanced part of the flow.

The initial conditions are,

$$\eta = \eta_0 \quad (a \text{ constant}) \quad \text{for} \quad |x| < L$$

$$\eta = 0 \quad \text{elsewhere} \quad \quad \quad \quad \quad @ \quad t = 0$$

$$v = 0, u = 0 \quad \text{everywhere}$$

This is a finite-width 'square wave' initial condition with a definite horizontal scale $L$, whereas Gill's problem of a step-function initial condition had no horizontal scale.

There are several ways to solve the problem: it is linear, so that two solutions can be added to give another solution; so most easily you can take Gill's steady solution (his Fig. 7.1) for a 'step' initial condition and add another step a distance $2L$ away of the opposite sign to make a square wave.

Other methods are to use the solution to problem 2.2 with Fourier analysis to add up sine-waves to make the square wave initial condition (but this requires experience with Fourier transforms); or, find solutions in each of the three regions and match then at $x = L$, and $x = -L$.

The solution for $\eta$ will be symmetric about $x = 0$.

- Calculate the ratio of potential energy $PE/KE$ of potential energy/kinetic energy as in Gill §7.2.3 Describe how the solution depends on the parameter $L/a$ (and on $g$ and $H$).