Solve and turn in 3 of the following problems; use the rest in studying for the take-home final test.

**Typo corrections and additions in this color.**

1. **Equatorial Kelvin waves.**
   
   Use the hydrostatic equations for a single layer of homogeneous (constant density) fluid to find a solution for Kelvin waves at the Equator. Write the x- and y- MOM (momentum-) equations for \( u, v \) velocities and \( \eta \) surface elevation, with the Coriolis frequency approximated by \( f = f(y) \) which is the first term in the Taylor series approximation to \( \sin(\text{latitude}) \). \( \beta = (2\Omega/a) \cos \theta \) so near the Equator we take \( \beta = 2\Omega/a \), which is constant). Unlike a mid-latitude \( \beta \)-plane, there is no \( \Omega_y \). Here we approximate the spherical Earth with Cartesian \((x,y)\) geometry.
   
   • Find a solution with the north-south velocity \( v = 0 \) everywhere. This solution will be wave-like in x- and t but you must solve for the y-structure of the wave: \( \eta = m(y) \exp(ikx - i\omega t) \)
   
   • Include the next term in the Taylor-series approximation to \( f(y) \), using \( \sin x = x - x^3/6 \) and find solve for the Kelvin wave.
   
   • These waves in the ocean are a significant part of seasonal response and el Niño/Southern Oscillation cycles. Interpret the result in terms of a ‘reduced gravity’ 1 ½ layer model with \( g \) replaced by \( g' \). Parallel \( \Delta \rho \) is the small density difference between the thin upper layer of depth \( H_1 \) and the deep, almost motionless lower layer with \( H_2 \Rightarrow \infty \). What is the north-south scale of the waves? (Try putting in numbers, for example \( H_1 = 100 \text{m}, \Delta \rho / \rho = 2 \times 10^{-3} \)). Equatorial Kelvin waves are also significant in the atmosphere, with correspondingly larger scale and faster propagation.

2. **The atmosphere/ocean heat engine: internal and external energy**
   
   Solar radiation drives almost all of the interesting motions of the atmosphere and ocean (there are gravitational tides of course). Consider a column of air small enough that it may be considered to have uniform temperature and density. The column is heated by an amount \( \delta T \). The top of the air column has a lid that is free to move vertically (it has a weight on it), thus the pressure in the gas does not change. • The thermal expansion coefficient of a fluid is defined as
   
   \[ \alpha \equiv -\rho^{-1} \partial \rho / \partial T \ \text{K}^{-1} \text{ at constant pressure } (K \text{ is the absolute Kelvin temperature}). \]

   • Find \( \alpha \) for air, as a function of its temperature \( T \) for the case of an ideal gas. Compare this with the thermal expansion coefficient for water which is roughly \( 2 \times 10^{-4} \ \text{K}^{-1} \).

   • What fraction of the heating goes into internal energy \( \mathcal{C}_V T \) and what fraction goes into gravitational potential energy, PE? For an ideal gas \( \mathcal{C}_V = 5R/2 \) and the specific heat at constant pressure is \( \mathcal{C}_p = 7R/2 \) where \( R = 287 \ J / \text{K}^3 / \text{kg} \). Now consider heating the top of the ocean with solar radiation: again imagine heating a uniform slab of water of thickness \( h \) by an amount \( \delta T \). Again, what fraction of the heating goes into gravitational PE and what fraction into internal thermal energy? The specific heat capacity at constant pressure for water is about \( 4000 \ J / \text{K} / \text{kg} \). The temperature is uniform throughout the water, and its upper surface is free to expand upward as it warms. Note the dependence of the result on the thickness of the layer being heated.

   • Compare the heat capacity of the oceans and atmosphere. The masses of ocean and atmosphere can be quickly compared by noting that the surface atmospheric hydrostatic pressure is 1 bar (1000 millibars) and in the ocean the pressure rises by 1 decibar (0.1 bar) for each meter of increasing depth.

   Comparing the two results will give you a glimpse of the different roles the atmosphere and ocean play in the climate system: one fluid dominates heat storage and the other dominates thermal expansion of the ‘heat engine’.

3. **Waveguides**
   
   We have emphasized the differences between dispersive and non-dispersive waves, in particular the way an isolated wave-pulse (like a bell-curve in shape) will remain compact and isolated in a non-dispersive wave system. Consider long gravity waves without Coriolis effects, in a channel with rigid side-walls at \( y = 0, y = L \). These are non-dispersive waves. Fluid extends to \( x = \pm \infty \) in this channel. Traveling waves reflect off the walls and can form standing waves. The boundary condition at these walls is \( v = 0 \)

   • Show that the boundary condition on the surface elevation is \( \eta_y = 0 \) at \( y = 0, L \).

   • Find wave solutions of the form \( \eta = m(y) \exp(ikx - i\omega t) \), solving for \( m(y) \) and the dispersion relation. You will find an infinite set of possible modes, in which the y-wavenumber takes on a set of well-determined values: eigenvalues.

   • Sketch the dispersion relation for several modes, and show that the group velocity of each mode, with respect to energy propagation in the x-direction, is now dispersive!

4. **Long, non-dispersive Rossby waves**
   
   When you find special limiting cases of a GFD wave or flow, it may have especially simple physical description. Rossby waves with a free upper surface become non-dispersive if their wavelength is long enough, with phase and group propagation directed westward. Make a physical model of these ultra-long Rossby waves as follows:
Consider a 1-layer homogeneous fluid with a resting depth given by \( h = H(1 - \gamma y) \), where \( \gamma \) is the slope of the bottom. Suppose that geostrophic motion of the fluid is in the \( y \)-direction (\( u=0 \)) and so slow that fluid columns are ‘rigid’ Taylor columns, for which \( h(y) + \eta(x,t) \) is constant following a fluid parcel (assume \( \eta \) does not vary in \( y \)). Write this as a ‘MASS’ conservation equation involving \( \eta \).

- Combine this rule with geostrophic balance to find a wave equation for \( \eta(x,t) \). Sketch the \( x-z \) shape of the wave and show graphically how it works. This is a model of Rossby waves, using the bottom slope instead of the \( \beta \)-effect. This is the same as approximating the potential vorticity \( q = (f + \zeta)/h \) as just \( f/h \). You can sketch the spherical fluid to see how the analogy works if the fluid columns preserve their height measured parallel with \( \Omega_G \), the rotation vector.

5. Spin-down of a geostrophic circulation by surface friction

A current in geostrophic balance varies sinusoidally in \( x \): \( v = A \cos k_0 x \) at time \( t=0 \). The fluid is homogeneous (constant density), with uniform depth \( H \).

- With an Ekman layer at the bottom which brings \( u \) and \( v \) velocities to zero at \( z=-H \), write the expression for the Ekman pumping, \( w(x) \) in terms of \( v \).
- Then write the interior vertical vorticity equation which will describe the evolution of \( v \) with time. First assume the upper surface is rigid, and \( w = 0 \) there. Solve for \( v(x,t) \). Note that you do not need to use the detailed Ekman profile once you know the relation between Ekman volume flux and geostrophic velocity. Note how the spin-down time varies with \( d/H \), the ratio of Ekman layer thickness to total depth.

- Allow the upper surface to be free, with Rossby radius \( (gH)^{1/2}/f \). Show how this affects the spin-down time, and suggest how the energy balance causes this change.

6. Mountain waves in a non-rotating, stratified fluid.

We showed the fundamental solutions for Rossby waves generated (a) by an oscillation point source of energy and (b) waves generated by a uniform zonal flow over a mountain. Instead of this propagation in \( (x,y) \), consider the \( (x,z) \) problem of internal gravity waves generated by flow over a mountain ridge, with no variation in the \( y \)-direction. These waves were introduced in the Fall Quarter fluids course. The internal wave equation for the vertical velocity \( w(x,t) \) is

\[
(w_{zz} + w_{xx})_t + N^2 w_{xx} = 0 \quad \text{(e.g., Gill § 6.4 - 6.5)}
\]

Recall the dispersion relation that follows:

\[
\sigma^2 = \frac{N^2 k^2}{k^2 + m^2} = N^2 \cos^2 \theta
\]

where \( k, m \) are the horizontal and vertical wavenumbers and \( \theta \) is the angle of the wavenumber vector \( (k,m) \) from the horizontal.

- By analogy with the Rossby wave problem, write the dispersion relation for internal gravity waves with a uniform horizontal mean flow \( (U,0) \). (A complete derivation is not necessary: use the idea of the Doppler shift in the \( x \)-direction to modify the dispersion relation).
- Look at the case \( \sigma = 0 \), that is stationary internal waves, generated for example by flow over a ridge. Describe the shape of the wave crests. Notice that as with Rossby waves there is a ‘peculiar’ solution \( k = 0 \), which can be thought of in the limit \( \theta \rightarrow \pi/2 \) (or \( k/m \rightarrow 0 \)). Show that, although these are waves with vanishing intrinsic frequency, they have finite strong group velocity and could propagate upstream. (This leads to a linear theory of ‘blocking’ of a stratified flow by a mountain ridge.)

7. Overturning circulations: a simple linear Hadley cell

Convection is one of the most difficult subjects in GFD. There are few simple solutions to start with, other than the basic instability of a fluid heated from below or cooled from above. However we often have a stably stratified fluid in which there is some radiative heating or cooling. Suppose there is a stable temperature gradient \( T_0(z) \). Write the temperature conservation equation for \( T(x,z) \) assuming a heating and cooling occurs with spatial form \( G(x,z) \)

\[
\frac{D T}{D t} = G(x,z) \quad (T = T_0(z) + T'(x,z))
\]

Neglect the nonlinear terms, keeping the dominant, linear advection of temperature.

Consider steady circulation of this fluid using a Boussinesq approximation, so that the flow is approximately non-divergent in \( x,z \). That is, \( u_x + w_z = 0 \). Define a stream function \( \psi(x,z) \) for the flow and find solutions, for example specifying

\[
G(x,z) = A \cos(k_0 x) \cos(k_0 z)
\]

How might the solutions behave if \( G \) did not have a zero-average in the \( x \)-direction (for example if we heat the fluid at one \( z \)-level and cool at another level)? How might diffusion of temperature affect the solution? (If \( G \) is a compact heat source in the middle of the fluid, the flow may reach far away from it. Internal gravity waves with low frequency and nearly horizontal velocity (nearly vertical wavenumber vector) act to set up this distant flow.)