Rossby waves

Rossby waves are the fundamental time-dependent motion of the large-scale oceans and atmosphere (and the atmospheres and possibly oceans of other planets). In a narrow sense they arise from the Earth’s rotation and spherical shape, but they are much more general. Mountains and slopes provide a background for topographic Rossby waves. Mean flows provide a vorticity environment for something like Rossby waves (which may often lead to instability of the flow). Turbulence stirs and mixes PV (potential vorticity) and can radiate Rossby waves.

Generally these waves have ‘steepness’ (a measure of their nonlinearity) which is not small: their phase speed is often no greater than typical fluid particle velocities. Thus they are rarely perfectly linear. An implication of this is that they interact strongly with the general circulation of the fluid.

The background PV field provides an ‘elasticity’ making fluid experiments in the GFD lab look more like ‘Jello’ than water. Rossby waves are undulations (wiggles) on the background PV gradient, and they can tear up and reshape the background PV. Long scales (long waves) feel strong elastic-like restoring forces while shorter scales (measured by the non-dimensional parameter $U/\beta L^2$) do not feel the elasticity and act as ‘geostrophic turbulence’. We principally use vorticity and PV dynamics to think about Rossby waves, because it is so difficult to know about the pressure field in nearly geostrophic flow. Pressure is active (and usually nearly hydrostatic and geostrophic) yet small departures from the geostrophic pressure are essential to the flow. By taking the curl of the MOM equations we get rid of some of its effects, keeping only the thermal wind related ‘baroclinic twisting’.

**Derivation from the vertical vorticity equation.** The full vorticity equation describes the ‘vector-tracer’ property in which we imagine marking the vorticity vector $\mathbf{\omega} \equiv \nabla \times \mathbf{u}$ on fluid particles with a dye ‘arrow’; as the fluid moves, with shear and strain, the dye arrow continues to show the vorticity vector as it translates, rotates and stretches. This is the same as the conservation of the ‘strength’ of a vortex tube, made up of a bundle of vortex lines (the vortex line is a curve which is everywhere tangent to $\mathbf{\omega}$). This is also the same as Kelvin’s circulation theorem which describes the rate of change of the vorticity normal to a ‘disc’ of fluid due to change in the area of the disc or change in its orientation. We can adopt this theory to a rotating Earth quickly, by taking account of the geopotential $\Phi \approx gz$, recognizing that surfaces with $\Phi=$constant are the ‘horizons’ of the planet, and replacing the vector vorticity by the total vorticity (relative + planetary), $\mathbf{\omega}_{\text{hrs}} = \mathbf{\omega} + 2\bar{\Omega}$.

We added stratification which, for an incompressible fluid contributes a new term

$$\rho^{-2} \nabla \rho X \nabla p$$
to the vorticity equation. This is the baroclinic production of vorticity. ‘Buoyancy twisting’ can
make fluid spin. An internal gravity wave can be imagined as a twisting of the fluid by tilted
surfaces of constant density, and the responding horizontal vorticity in the wave. The lucky
result is that all our previous conservation principles for vorticity still hold in the special
direction normal to surfaces of constant density, since \( \nabla \rho \times \nabla p \) makes no contribution to the
vorticity component normal to those surfaces. \( \nabla \rho \times \nabla p \) is a nearly horizontal vector that lies
parallel to surfaces of constant density or pressure. Under hydrostatic scaling \((H/L)^2 << 1\) (H is
the vertical scale of the motion and L the horizontal scale), both \( \nabla p \) and \( \nabla \rho \) are nearly vertical,
so baroclinic twisting creates mostly horizontal vorticity. The much smaller vertical vorticity, \( \zeta \),
obeyes its own key equation, with density appearing mostly as a control on vertical velocity, but
not through direct twisting. (Note that the ratio of vertical vorticity/horizontal vorticity \( \sim u_y/u_z \)
\( \sim H/L \) which is small).

The build-up to long waves and Rossby waves runs through Vallis’ chapter 4 and 5. In the
simplest case of a thin shell of fluid on a sphere, consider a disc of fluid moving two-
dimensionally without changing its horizontal area \( A \). Kelvin’s theorem with Earth’s rotation
(Vallis 4.4.1) for example shows succinctly that

\[
\frac{D}{Dt} (C + 2\Omega A_{\text{perp}}) = 0 \quad (C \text{ is the circulation, for a small disk } C = \zeta A)
\]

\[
\frac{D\zeta}{Dt} = -\frac{\Omega}{A} \frac{DA_{\text{perp}}}{Dt} = -2\Omega \frac{D(sin \theta)}{Dt} = -\frac{2\Omega \cos \theta}{a} \nu
\]

\[
\equiv -\beta v \quad \text{or}
\]

\[
\frac{D(f + \zeta)}{Dt} = 0
\]

Here \( A_{\text{perp}} = A \sin \theta \) is the area \( A \) projected on the plane of the Equator. \( \theta \) is the latitude.
There is some subtlety here in assuming \( A = \text{constant} \), because heretofore we have said that fluid
is ‘stiffened’ in the direction parallel with \( \hat{\Omega} \). That would tend to allow \( A \) to change as fluid
moves north or south…but file this technical issue away for a later time. The procedure here
works because the thin shell very much constrains the fluid to move nearly horizontally.

The \( \beta \)-plane approximation. We adopt the mid-latitude \( \beta \)-plane in which we take the
Coriolis frequency \( f \) to vary with latitude: \( f = f_0 + \beta y \). This is an approximation to the exact
vertical component of planetary vorticity, \( 2\Omega \sin(\text{latitude}) \). Carl Gustav Rossby introduced the
\( \beta \)-plane to explain the wavy flow of the westerly winds…often stationary waves whose phase
propagation westward is cancelled by eastward zonal flow. Neglecting the spherical geometry
leads to significant errors of order \( L/a \) where \( L \) is the length scale of the waves and \( a \) the radius
of the Earth. Yet, this approximation is still widely used because it so greatly simplifies the
math. There is also a potential problem with the neglect of the horizontal component of the Earth’s rotation vector in the MOM equations, but this leads to errors mostly near the Equator, provided that $H/L << 1$. It is known as the ‘traditional approximation’ because everyone does it yet few understand it. The $\beta$-plane is further described below.

Thus with constant depth $H$ we arrive at the vertical vorticity equation as in Vallis 5.1, 5.3, in the form

$$\frac{D}{Dt} (\nabla^2 \psi + \beta y) = 0$$

$$\nabla^2 \psi_t + \beta \psi_x = 0 \quad (if \ U / \beta L^2 << 1)$$

where the second equation has been linearized. The approximations to get to this point are that $Ro<<1$ ($Ro=U/fL$);

$(H/L)^2 <<1$ (hydrostatic);

$1/fT <<1$ (the other Rossby number);

$L/a <<1$ ( $\beta$-plane approximation)

and $E <<1$ where $E = \nu/fH^2$. $E$ is the Ekman number based on the kinematic viscosity $\nu$ and it relates to Ekman layer friction. ($E^{1/2}$ is the ratio of the thickness of the Ekman boundary layer at the base of the fluid, and the height scale of the motion, $H$).

Finally the linearized vorticity equation assumes that the wave steepness,

$U/(L/T) \equiv U/c_p \ll 1$,

which is the ratio of the fluid velocity in a wave motion and the phase speed $c_p \sim L/T$. Note, $c_p$ and $U$ are not the same, though it can be confusing.

The background PV (potential vorticity) field. The Earth’s spherical shape and its complex topography, both on land at the sea-bed, provide a background PV field that makes possible Rossby waves. The waves are undulations of this basic field of $f/h$, just as internal gravity waves are undulations of a basic density stratification. In addition, the time-averaged mean circulation provides a background PV field through both its relative vertical vorticity $\zeta$ and its effect on layer thickness (principally in the stratified fluid, with baroclinic wave dynamics).

Thus maps of the mean PV are key to dynamics. When stratification is added we have the $f/h$ maps of isopycnal (constant potential density-) layers, giving a 3-dimensional rendition of the PV field. Many new effects then occur, especially baroclinic instability, interaction between waves and mean circulation, baroclinic Rossby modes; all of these understandable in terms of PV dynamics.
The $f/h$ contours (‘geostrophic contours’) of the Atlantic Ocean are shown in Fig. 5.0. Rossby waves would appear as gentle undulation of these contours (recall, PV is a tracer that moves with the fluid). If the waves are ‘steep’, that is nonlinear, the PV contours will begin to form closed maxima or minima, and the background PV gradient gets ‘confused’.

Fig. 5.0  Geostrophic contours (mean PV-contours, $f/h =$ constant) for the N. Atlantic Ocean. (Rhines, *J.Fluid Mech.* 1969). Barotropic Rossby/topographic waves can be visualized as wiggling of these contours. Where the depth is uniform these are just latitude circles, but they deflect equatorward over the mid-ocean ridge and at the continental slope topography near the boundaries. These provide ‘wave-guides’ for Rossby waves/topographic Rossby waves. Free, slow circulation conserving PV on fluid particles would move along these contours.

Note: The map in Fig. 5.0 is relevant to barotropic dynamics, as if the fluid were unstratified. It must be augmented by baroclinic maps of $f/h_{dens} =$ constant where $h_{dens}$ is the thickness of a fluid layer between two nearby surfaces of constant potential density. This simply follows from the stratified (Ertel-Rossby) expression for PV. These $f/h_{dens}$ contours relate to circulation gyres at each vertical level (or on each isopycnal surface). Barotropic PV and baroclinic PV are still be reconciled by the research community. Maps of PV for the stratified
oceans were first given by McDowell et al., *J. Phys. Oceanogr.* 1982. In this case the ‘background’ PV is no longer just the solid Earth, but it is the mean ‘topography’ of isopycnal layers. Since PV is conserved following the fluid motion (until dissipated by friction or otherwise externally forced), it is a dynamic tracer of water masses in the oceans…a tracer that distorts and transports itself! PV encodes the general circulation through the relation between vorticity and velocity.

*Alternate derivation of linearized wave-equation.* Vorticity principles formed the derivation of the vertical vorticity equation, written for geostrophic flows in terms of the free-surface height, \( \eta \), above. Another connection can be made with the familiar long-gravity-wave equations, and it also provides a relatively quick derivation of wave equation. We assume that nonlinear terms can be neglected in both momentum and mass-conservation equations. Making the hydrostatic-pressure approximation, (with or without the small Rossby-number assumption used earlier), the horizontal velocity is independent of \( z \), and the momentum equations become

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v & = -g \eta_x \\
\frac{\partial v}{\partial t} + f u & = -g \eta_y
\end{align*}
\]

(5.1) MOM

MASS-conservation becomes

\[
\eta_t = -[(hu)_x + (hv)_y].
\]

(5.2)

where \( h = H + h'(x,y) + \eta(x,y,t) \). This is derived by noting that \((hu, hv)\) is the lateral volume transport across a vertical section of the fluid, and flow divergence must be balanced by vertical motion at the free surface.

Fig. 5.1 Mass conservation relates horizontal variations of \( hu \) to vertical movement of the free surface.

Alternatively, with \( u \) and \( v \) independent of \( z \),

\[
\begin{align*}
u_x + v_y + w_z &= 0
\end{align*}
\]

we multiply by \( h \), and use the boundary conditions \( w=\eta_t \) at \( z=0 \) and \( w=-uh_x-vh_y \) at \( z=-h \) to find the same result. Forming \((h(5.1a)_x+(h(5.1b)_y\) which is known as a ‘divergence’ equation, describing the rate of change of the horizontal divergence of volume flux, \( \nabla \cdot (hu) \). Then form
-(h(5.1a)y + (h(5.1b))x which is a vorticity equation. These combine with +(1/g)(h(5.2),) + (h(5.2)y) to give

\[ \frac{\partial}{\partial t} [\eta_t + f^2 \eta - \nabla \cdot gh \nabla \eta] + gfJ(\eta, h) - ghJ(\eta, f) = 0 \]

(e.g. Pedlosky, Geophysical Fluid Dynamics, 3.7). Here, as described earlier, J is the Jacobian operator, written in various ways as

\[ J(a,b) = a_y b_x - a_x b_y = \nabla a \times \nabla b \cdot \hat{z} = \frac{\partial (a,b)}{\partial (x,y)} \]

This equation may seem strange, but it is useful, because it incorporates in one equation long gravity waves, Kelvin waves, Rossby waves due to the \( \beta \)-effect and topographic Rossby waves due to a sloping topography (the next-to-last term). It makes it clear that the high-frequency waves and the low-frequency Rossby-type waves can emerge from one derivation. It is the ‘fast’ first term \( \eta_{ttt} \) which gives us gravity or Kelvin waves.

**Scale analysis of the wave equation.** Eqn. 5.1 describes a wealth of waves, both high-frequency and low-frequency. If the wave frequency is much greater than \( f \), the terms in the first set of brackets, with \( f = 0 \), describe long, hydrostatic, non-dispersive gravity waves. With \( h = \text{constant} \), \( f \neq 0 \), Coriolis effects make these waves dispersive and provide a low-frequency cut-off at \( f \). More generally, if \( T \) is the time-scale of the waves (\( T = 1/\text{frequency} \)), and \( L \) is their horizontal length scale (\( L = \text{wavelength}/2\pi \)), and \( H \) the mean depth, then scale analysis of the five terms in the equation gives, respectively:

\[ \frac{\eta}{T^3}, \frac{f^2 \eta}{T}, \frac{gH \eta}{L^2 T}, \frac{H \delta g \eta}{L^2}, \frac{gH \beta \eta}{L} \]

where \( H \delta \) is the typical size of topographic mountain heights at the scale \( L \). One can see that a new set of motions with low frequency may be possible by neglecting the 1st term. This is works if \( (L/T) \ll (gH)^{1/2} \), the phase speed of simple gravity waves. If we balance the relative vorticity change, term 3 with terms 4 or 5 the frequency will be \( T^{-1} \sim f \delta \) or \( \sim \beta L \).

For this choice of frequency range, the first term divided by the second term is just \( O(\delta^2) \). Evidently, if the height of the topography (measured over the horizontal wave-scale \( L \)) is much smaller than the mean depth, then the first term is negligibly small, leaving the equation

\[ \frac{\partial}{\partial t} [f^2 \eta - \nabla \cdot gh \nabla \eta] + gfJ(\eta, h) - ghJ(\eta, f) = 0 \]

For a uniform depth fluid, \( h = H \), a constant. Then we recover the Rossby wave equation,

\[ \frac{\partial}{\partial t} [\nabla^2 \eta - \frac{f^2}{gH} \eta] + \beta \eta = 0 \]
Remembering that $\eta$, $p$ and $\psi$ are all simply proportional to one another in the simplest geostrophic flows with a single layer and free surface, this is just the same as
\[ (5.2b) \quad \nabla^2 \psi_t - L_D^2 \psi_t + \beta \psi_x = 0 \quad L_D = (gH)^{1/2} / f \]
Recall how the PV for a layer of homogeneous fluid is $q=(f+\zeta)/h$, and that with very slow flows, this is approximately $f/h$. Thus the ‘background’ PV field provided by the Earth involves both variations in $f$ and variations in $h$. We can have Rossby waves due to the one or topographic Rossby waves due to the other.

**Topographic Rossby waves (optional section).** Let us choose a particular bottom topography, a simple up-slope to the north. Let
\[ h = H + h'(x,y) + \eta(x,y,t) \]
with $h' = -\alpha y$. For small amplitude waves, $\eta$ may be neglected in the expression for $h$. Then Eqn. 5.2a becomes
\[ -(f^2/g)\eta_t + h(\eta_{xx} + \eta_{yy}) + (h'/h)\eta_x - fh\eta_x = 0 \]
\[ -(f^2/g)\eta_t + (H - \alpha y)(\eta_{xx} + \eta_{yy} - (\alpha f(H - \alpha y))\eta_x) + f\alpha\eta_t = 0 \]

Scale analysis of this equation gives the following terms:
\[ \frac{f^2\eta}{gT} \quad \frac{(H + \alpha L)\eta}{L^3 T} \quad \frac{(H + \alpha L)\eta}{L^2 T} \quad \frac{\alpha\eta}{LT} \quad \frac{\eta}{L^2 T} \quad \frac{f\alpha\eta}{L} \]
Earlier we saw that the topographic height divided by mean depth, $\delta (= \alpha L/H)$ had to be small to achieve a balance involving the new topographic term. So here we neglect the small $O(\alpha L/H)$ parts of terms 2 and 3, and similarly neglect term 4 relative to terms 2 and 3, giving
\[ (5.3) \quad (\eta_{xx} + \eta_{yy}) - (f^2/gH)\eta_t + (f\alpha/H)\eta_x = 0 \]
which is the equation for topographic Rossby waves. Its most important property is that it is linear: two solutions added together give a third solution, and initial conditions with a complicated shape in $x$ and $y$ can be expressed as a sum of many Fourier components. (5.3) has constant coefficients and thus has simple solutions for propagating plane waves.

These topographic waves are a key part of coastal ocean dynamics. Their phase propagation is in the same direction as Kelvin waves, for the usual sloping bottom of the continental rise, slope and shelf (this could be called ‘pseudo-westward’, in memory of Rossby waves. In fact Kelvin modes and topographic modes become intermingled with realistic bottom topography.

**Some Mathematical Background.** How does this equation compare with those of classical physics? We are taught that linear second-order pde’s (partial differential equations) fall into three groups, hyperbolic (e.g.,
the classic wave equation), elliptic (e.g., Laplace’s equation) and parabolic (e.g. the heat-diffusion equation). The classic wave equation for vibrating strings and membranes, long gravity waves on water and sound waves is hyperbolic with respect to space and time,
\[ \eta_{xx} + \eta_{yy} - c_0^2 \eta_t = 0. \]

Wave-like solutions \( \eta = \text{Re}(A \exp(ikx + ily - i\sigma t)) \), after substitution in the equation, yield the dispersion relation
\[ \sigma^2 = c_0^2 (k^2 + l^2) \]
for which the phase speed, \( \sigma / (k^2 + l^2)^{1/2} \) is independent of both wavelength and direction of propagation. That is, the waves are both non-dispersive and isotropic. The equation has characteristic curves, along which solutions propagate. Isolated, pulse-like initial conditions (a stone thrown in a pond) are prevented from dispersing into sinusoidal waves...Fourier components. A top-hat like wave can propagate without change of shape along a vibrating string; in two-dimensions it will decay in amplitude as it spreads out in a ring (leaving a narrow wake). The stone thrown in a pond of course excites short gravity waves and even shorter surface-tension waves (ripples), which disperse into sine-wave trains. Text-books on waves are usually divided neatly into two sections: dispersive and non-dispersive waves, because the solutions and techniques for solving them differ so greatly (e.g., Lighthill, \textit{Waves in Fluids}, Cambridge 1977, Whitham, \textit{Linear and Nonlinear Waves}, Addison-Wesley 1974).

Eqn. 5.3 does not fall neatly into one of these three categories, because it is formally a third-order pde. However here (and with the classical wave equation), the time-variable is separable. Assuming sinusoidal time-dependence the remaining equation is of 2d order and elliptic in space variables \( x \) and \( y \). This too occurs when an \( \exp(-i\sigma t) \) factor is separated from the classic wave equation, leaving the Helmholtz equation,
\[ \nabla^2 \eta + (c_0^2 / \sigma^2) \eta = 0. \]

Elliptic pde’s require boundary conditions ‘all-around’, for example on a boundary surrounding the fluid, for the solutions to be well-determined. Solutions however are readily found, both plane waves and for waves generated at a single point in space. These latter are ‘Green functions’, a sort of ‘impulse response’ for the wave equation, and are useful in building intuition for the cause and effect in the fluid.

\textit{Rossby wave solutions.} Consider solutions to 5.2b in the form of a plane wave,
\[ \psi = \text{Re}(A \exp(i k x + i l y - i \sigma t)) \]
\[ = \text{Re}(A \exp(i \vec{k} \cdot \vec{x} - i \sigma t)) \]
\[ = \text{Re}(A \exp(i \xi(x,y,t))) \]
‘\( \text{Re}(\_\_) \)’ denotes the real part. The \textit{wavevector} \( \vec{k} = (k,l) \) is perpendicular to the wave crests, which are lines of constant \textit{phase}, \( \xi(x,y,t) \). Substituting in the wave equation, all the physics boils down to the \textit{dispersion relation}, connecting the wavenumbers and frequency.

\[ \sigma = -\frac{\beta k}{k^2 + l^2 + 1/L_D^2} \]
where \( L_D = (gH)^{1/2}/f \) is the Rossby deformation radius. Note that \( L_D \) is equal to \( c_0/f \) where \( c_0 \) is the propagation speed of long gravity waves (with \( f=0 \)); this definition of the ‘Rossby radius’
turns out to be true under much more general circumstances. Rossby waves are approximately *transverse*: the velocity is along the wave crests, perpendicular to the wave vector \((k,l)\). This is suggested by geostrophic balance (because the pressure gradient is parallel with \(k\)). Or, the mass-conservation equation \(u_x + v_y = 0\) also suggests the same thing, because when \(u\) and \(v\) are expressed in terms of the wave-like \(\eta\), we have \(ik \cdot \vec{u} = 0\), again saying that the velocity lies along the wave-crests (to order \(\sigma/f\)).

Note the same result holds for *topographic Rossby waves* if we replace \(\beta\) by the \(f\alpha/H\) where \(\alpha\) is the slope of the bottom, here taken to be \(h = H - \alpha y\).

We draw the dispersion relation, \(\sigma\) as a function of \(k\) and \(l\) (Fig. 5.2). It is shaped like a ‘witch’s hat’, peaking near the origin. Height contours of \(\sigma(k,l)\), show possible wave-vectors for a constant \(\sigma\).

![Dispersion relation](image)

_Fig. 5.2_ Frequency as a function of wavenumbers \((k,l)\) for Rossby waves.

*Waves in a channel.* A complete mathematical problem consists of both equation and boundary conditions. Here we add vertical walls to the topographic wave problem, at which the normal fluid velocity must vanish. Thus suppose

\[ v = 0 \quad \text{at } y = 0, L. \]

The momentum equations show that \(v = (g/f)\eta_x + O(1/fT)\), so we will simply require that \(\eta_x\) vanishes at the two east-west running boundaries of the channel. This will be achieved by the separable solution

\[ \psi = A \sin \lambda y \exp(ikx - i\sigma t) \]

if we choose \(y\)-wavenumbers, \(l = n\pi/L\), for \(n = 1, 2, \ldots\). The dispersion relation becomes
\[ \sigma = \frac{k}{k^2 + n^2 \pi^2 / L^2 + 1 / L_D^2} \]

The frequency rises to a maximum at \( k^2 = n^2 \pi^2 / L^2 + 1 / L_D^2 \), falling toward zero for both longer and shorter waves. The curve is a cut through the two-dimensional figure, 5.2, at fixed north-south wavenumber, \( l \). Notice that the wave pattern always moves toward negative \( x \) (westward in this case), for \( \sigma/k < 0 \). However the group velocity along the channel, \( \partial \sigma / \partial k \), can take either sign; it is the slope of the \( \sigma(k) \) curve for a fixed value of \( n \). Energy propagates westward in the longer waves and eastward in the shorter waves.

There are several interesting limiting cases. For the shorter waves (that is, with large wavenumber)

\[ \sigma \approx -\beta/k; \]

the frequency varies inversely with \( k \). For the longest waves, as \( k \to 0 \),

\[ \sigma = \frac{-\beta}{n^2 / L^2 + L_D^{-2}} k \]

the waves are non-dispersive. All waves in this limit propagate at the same speed, westward along the channel. These waves have \( k \ll l \), so that the gradient of the free-surface height, proportional to \( \nabla \psi \), points nearly across the channel (north/south), and the oscillating velocity field is directed nearly along the channel, \( u \gg v \). For this example we have kept the Rossby
Radius, $L_D$, large compared with the channel width, $L$. If this were not so we would have a larger regime of non-dispersive waves moving west along the channel, in fact all wavenumbers obeying $(k^2 + l^2)^{1/2} L_D << 1$. Eqn. 5.3, for this case, $L >> L_D$, simplifies to a remarkably simple 1st order pde:

\begin{equation}
\psi_t + (\beta L_D^2) \psi_x = 0
\end{equation}

The general solution of this wave equation is westward propagation without change of shape:

$$\psi = m(x + (\beta L_D^2)t),$$

for an arbitrary function $m(\bullet)$.

How to construct the wave pattern from an oscillating compact source. If a ‘wavemaker’ like a glass cylinder oscillating in and out of the water in the GFD lab experiment exists, with a single frequency, that should generate all the wavenumbers possible for that frequency. The size and shape of the wavemaker may favor some regions of wavenumber space. To construct the wavecrests (the curves of constant phase), use the wavenumber locus below (Fig. 5.4). Pick a single wave, draw its wavevector $k$ from the origin, noting that the wavecrests are perpendicular to $k$. Then draw the group velocity $c_g$ perpendicular to the locus: here that means pointing toward the center of the circle. So that wave component is seen in the direction of the group velocity. Go to the $(x,y)$ plane and sketch the wave packet there: go from the origin in the direction of $c_g$, a distance proportional to the magnitude of $c_g$. 
Fig. 5.4a Locus of possible wavenumbers at a single frequency, barotropic Rossby waves, rigid lid. The thin arrows are two particular waves, and the heavy arrows are the corresponding group velocity. The group velocity is the gradient of the surface $\sigma(k,l)$. Note that the long waves (small wavenumber) have larger group velocity than the short waves (large wavenumber).
You can see the form of the wavecrests begin take shape and sense that the longer waves move their energy more quickly (the ones moving west, northwest, southwest). There is an exact solution to this problem which shows that connecting the dots gives us wave crests that are parabolas, which can be written \(x + r = \text{constant}\).

Optional section: The exact solution for this ‘Green function’ is

\[
\nabla^2 \psi_i + \beta \psi_x = \delta(x, y) \exp(-i\sigma t) \\
\psi = \exp(-i\kappa x - i\sigma t) H_0^{(2)}(\kappa r); \quad \kappa \equiv \beta / 2\sigma \\
\approx (2/\pi \kappa r)^{1/2} \exp(-i\kappa(x + r) - i\sigma t) \quad \text{for } \kappa r >> 1
\]

where \(H_0^{(2)}(\kappa r)\) is a Hankel function (complex Bessel function) of the 2d kind and \(\delta(x, y)\) is the Dirac delta function: think of it as a Gaussian bell curve made ever narrower and ever taller so as to preserve the volume beneath it. Here \(r\) is the radius, \((x^2 + y^2)^{1/2}\). The solution decreases in amplitude with increasing \(r\), like \(r^{-1/2}\) as the wave energy is spread over a larger area. As suggested above, the phase is constant along parabolas in \((x,y)\) space, given by \(x + r = \text{constant}\). The wave crests sweep westward with time and form nearly zonal alternating currents to the west of the forcing. The energy flux is equal to the energy density \(X\) the group velocity, and it is...
the same in all directions. Yet, because the group velocity is so much smaller east of the forcing, the KE is largest there. A surface plot is shown below, Fig. 5.4c.

**Fig. 5.4c** Full solution for Rossby waves generated by an oscillating wavemaker, perspective view from the southwest. The parabolic wavecrests are visible. Contours are shown on the plane below.

**Rossby waves on a spherical Earth (optional section).** We now return to the more general case of a spherical Earth (of course the Earth has an equatorial bulge, and hence more closely approximates an ellipsoid with complex topography superimposed on it. Given that the smoothed geoid deviates from a sphere by about 21 km compared with a mean radius, $a$, of 6380 km. we shall neglect the difference). The momentum equations in spherical coordinates ($\lambda, \varphi, r$) are similar to their Cartesian $(x, y, z)$ counterparts. Notation varies among different texts but here we side with Gill, and let $\lambda$ be longitude (positive eastward), $\theta$ be latitude (positive northward) and $r$ radius (equivalent to $a + z$). Corresponding velocity components are $(u, v, w)$. They are derived in Gill, secs. 4.12, 11.2. Ignoring nonlinear terms, and setting the vertical velocity, $w$, to zero for this homogeneous fluid we have (similar to Gill sec. 11.2.1-.4),

\[
\begin{align*}
\frac{\partial u}{\partial t} - 2\Omega \sin \theta \: v &= -\frac{g}{a \cos \theta} \frac{\partial \eta}{\partial \lambda} \\
\frac{\partial v}{\partial t} + 2\Omega \sin \theta u &= -\frac{g}{a} \frac{\partial \eta}{\partial \theta} \\
\frac{\partial \eta}{\partial t} + \frac{1}{a \cos \theta} \left( \frac{\partial}{\partial \lambda} (hu) + \frac{\partial}{\partial \theta} (hv \cos \theta) \right) &= 0
\end{align*}
\]

(5.6)

Here $\frac{1}{a \cos \theta} \frac{\partial (\bullet)}{\partial \lambda}$ plays the role of $\frac{\partial}{\partial x}$ because as the constant longitude meridians converge towards the pole, the scale factor $(r \cos \theta)$ relates longitude to east-west distance. Similarly the $(r \theta) = y$, the north-south
distance. We have made the ‘thin-shell’ approximation, replacing \( r \) by its average value \( a \). The vertical momentum balance is assumed hydrostatic as usual; this neglects a Coriolis term proportional to north-south velocity, but it is very small unless one is close to the Equator. In Batchelor’s *Introduction to Fluid Dynamics*, Appendix 2 gives a very useful discussion of coordinate systems and corresponding vorticity and divergence expressions. The easiest way to derive these expressions is to draw a small area element bounded by constant circles of latitude and longitude, and work out the volume inflow through the faces of this area (for divergence) and (for vorticity) the circulation \( \vec{u} \cdot d\vec{l} \) integrated around its edges (where \( d\vec{l} \) is a vector line segment marking the boundary of this elemental area). This integral is equal to the product of area of the element and vertical vorticity, by Stokes’ theorem.

For simplicity consider the time-mean depth \( h \) to be constant, thus eliminating the topographic Rossby wave effect. The curl of the momentum equations gives

\[
\begin{align*}
\zeta_t + (f/a \cos \theta)(u_x + (v \cos \theta)_\phi) + \beta v &= 0 \\
\zeta &= (1/a \cos \theta)(v_\phi - (u \cos \theta)_\theta)
\end{align*}
\]

(5.7)

\[
=g/2\Omega \sin \theta \Delta \eta \quad \text{where}
\]

\[
\Delta \eta \equiv \left\{ \frac{1}{a^2 \cos^2 \theta} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2 \cos \theta} \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial \theta} \right) \right\} \eta
\]

Recognize the second term in eqn. 5.7a as the horizontal divergence \((u_x + v_y)\) in Cartesian coordinates). Using the mass conservation equation 5.6c, the vorticity equation becomes

\[
\frac{\partial}{\partial t} \Delta \eta - \left( \frac{(2 \Omega \sin \theta)^2}{gH} \right) \frac{\partial \eta}{\partial t} + \frac{2 \Omega}{a^2} \frac{\partial \eta}{\partial \lambda} = 0,
\]

(5.8)

which may also be written

\[
\frac{\partial}{\partial t} \Delta \eta - \left( \frac{f^2}{gH} \right) \frac{\partial \eta}{\partial t} + \frac{\beta}{a \cos \theta} \frac{\partial \eta}{\partial \lambda} = 0
\]

where \( f = 2\Omega \sin \theta \), and \( \beta = 2\Omega \cos \theta / a \) is the northward derivative of Coriolis frequency, \( f \). Here the horizontal Laplacian in spherical coordinates is written \( \Delta \eta \). This is the close analogue of eqn. 5.2a,b, for \( \beta \)-plane Rossby waves or topographic Rossby waves over a plane bottom slope, as the second version of the equation shows plainly. The full spherical problem, though complicated by the geometry, has a simple physical essence. Both topographic Rossby waves and ‘true’ Rossby waves originate in the conservation of potential vorticity for a homogeneous fluid. Eqn. 5.8 has elegant solutions on the sphere, described by Longuet-Higgins (Proc.Roy.Soc. London, A, 1964).

*The \( \beta \)-plane.* As described above, For Rossby waves with length scale \( (k^2 + l^2)^{-1/2} \) smaller than the Earth’s radius, we can approximate the spherical coordinates by Cartesian coordinates in a plane tangent to the sphere. This was Carl-Gustav Rossby’s brilliant insight, in 1939, which gave meteorology a strong and useful theory of the waves in the westerly winds. Thus writing \( dy = a \, d\theta \) and \( dx = a \, \cos \theta \, d\lambda \), we return to exactly equation 5.3,
\( (\eta_{xx} + \eta_{yy}) - \left(f^2/gH\right)\eta_z + \beta\eta_z = 0 \)

where \( \beta \) takes over the role of the bottom slope, \( \frac{f\alpha}{H} \). This is the \( \beta \)-plane equation for Rossby waves. It has two forms: for the \textit{mid-latitude} \( \beta \)-plane, \( f \) is taken to be constant except when differentiated in \( y \); \( f_0 \) is equal to \( 2\Omega \sin \varphi_0 \), where \( \varphi_0 \) is the central latitude of the region of interest. \( \beta \) is likewise taken to be constant, \( 2\Omega \cos \varphi_0/a \). Errors in making this approximation are typically of order \( L/a \). For the \textit{equatorial} \( \beta \)-plane, our origin for \( y \) is the Equator itself, and we take

\[ f \approx \beta y = 2\Omega y/a \]

and \( \beta = 2\Omega/a \), again a constant. Notice that now the vorticity equation has a non-constant coefficient, proportional to \( y^2 \). This makes it a Schrodinger equation as found in quantum mechanics, and suggests ‘potential well’ solutions known as the parabolic oscillator. Indeed, the Equator is a wave-guide not unlike the channel model derived for eqn. 5.3. Waves propagate east and west along the Equator while being trapped north-south by the increase of the Coriolis frequency, with latitude. The equatorial \( \beta \)-plane is a rather more accurate (\( O(L/a)^2 \)) approximation than the mid-latitude \( \beta \)-plane, which is a reason for being familiar with the full spherical form of the equation. We should note that the equatorial region has a crucially important Kelvin-wave mode which is missed by this low-frequency analysis, yet can be recovered by going back to the equatorial version of equation 5.1; see Gill, 11.4 (p. 434).

**Summary of Rossby wave properties.** The waves have these properties:

- They are dispersive (the phase speed varies with wavelength), although waves much longer than the Rossby Radius, \( L_D \), are non-dispersive, \( \sigma \approx -\beta L_D^2 \), and these propagate straight westward.
- They are anisotropic (the phase speed and frequency vary with direction of propagation, even for a fixed wavelength).
- They have wavecrests with phase speed moving \textit{westward} along latitude circles, \( c_p = \sigma/k < 0 \). This also describes the topographic waves calculate above, where the topography slopes upward to the north; adjust accordingly for other orientations of the slope, so that the wavecrests always move with shallower water to their right (reverse this in the southern hemisphere). See Figs. 5.4, 5.5, 5.6
- energy propagation, with group velocity

\[ \vec{c}_g = \left( \frac{\partial\sigma}{\partial k}, \frac{\partial\sigma}{\partial l} \right) \]

can occur in any direction. The group velocity, \( \vec{c}_g \), is the gradient of the surface \( \sigma(k,l) \), hence is perpendicular to the contours of constant frequency, pointing toward higher values of \( \sigma \).
- For east-west propagation, the group velocity is directed eastward for shorter waves and westward for longer waves. This is evident in the ‘Green function’ solution for waves generated by a point source, oscillating at a single frequency.
• With a rigid lid, \( L_D \rightarrow \infty \), the dispersion relation can be written
\[
\sigma = \beta \cos \gamma / K
\]
where \( K^2 = k^2 + l^2 \) is the total wavenumber and \( \gamma \) is the direction of phase propagation
\( (k = K \cos \gamma, l = K \sin \gamma) \). This use of polar coordinates in wavenumber space shows clearly the anisotropy (\( \gamma \)-) and wavelength dispersion (\( K \)-) effects.

• The magnitude of the group velocity is
\[
|c_g| = \sigma / k \equiv \beta / K^2 \text{ where } K^2 = k^2 + l^2.
\]
This coincidence (that the westward phase speed is also the magnitude of the vector group velocity) can be worked out from (5.9)

• The direction of the group velocity is \( 2\gamma \) (which can be seen from the plot of constant frequency in \( k,l \) space).

• They are ‘vorticity waves’ which are nearly non-divergent (the horizontal divergence, \( u_x + v_y \) is small, \( \text{Ro} \) compared with vorticity \( \zeta = v_x - u_y \)). Parcel arguments for Rossby waves are discussed below (Fig. 5).
  • they are nearly geostrophic, \( u_t \ll fv \) or \( g \eta_x \) or \( f \psi_x \).
  • their frequency increases with wavelength

• \( \eta \) is closely proportional to pressure, \( p (= \rho g \eta) \), and to the stream-function, \( \psi (= g \eta / f) \) for the horizontal velocity

• the vertical vorticity, \( \zeta = \nabla^2 \psi = (g/f) \nabla^2 \eta \)

• the horizontal velocity is related to the free surface slope according to (5.1),
\[
\Box \quad u = g \frac{-f \eta_y + i \sigma \eta_x}{f^2 - \sigma^2} \quad ; \quad v = g \frac{f \eta_x + i \sigma \eta_y}{f^2 - \sigma^2}
\]
The high- and low-frequency limits of these expressions speak for themselves.

• the ratio of potential energy, \( \frac{1}{2} g \eta^2 \), to horizontal kinetic energy, \( \frac{1}{2} \rho (u^2 + v^2) \), varies as \( L^2 / L_D^2 \), \( L \) being the horizontal length scale.
Fig. 5.5. Space-time (‘Hoevmueller’) plot of meridional velocity against longitude and time, from the MODE-73 experiment in the western Atlantic near 30N. Time progresses downward, and the phase of the 100km scale eddies moves westward at about 5 cm sec⁻¹. Rhines, The Sea, vol VI, 1977. This was the first observation of Rossby waves in the ocean.
Figure 5.6. Satellite altimeter observation of the sea-surface height field in the North Pacific, Chelton, 1995, also as time-longitude plots. Note the well-defined westward propagation speeds, which increase with latitude. Time progresses upward, and we are watching over 8 years. These waves are slow! They are most likely nonlinear Rossby waves in the upper ocean, with the thermocline acting as a ‘free surface’ with a reduced gravity, so that $L_D \approx 50$ km or so. A 1 ½ layer model seems to describe them semi-quantitatively, and this is the non-dispersive,
long-wave limit, which gives a simple westward phase and energy propagation, varying with latitude as \( \beta \) varies.

Waves on a zonal current. (Vallis 5.7) If we add a uniform zonal flow, \( \mathbf{u} = (U_0,0) \) to the wave equation, it adds a term \( U_q \). Generally the advective acceleration is \( \mathbf{\tilde{u}} \cdot \nabla q \) and it includes both linear and nonlinear terms. Let \( \mathbf{u} = U + u' \), and \( q = Q + q' \) describe the wave-related quantities \( u' \) and \( q' \) and the mean-state, \( U \) and \( Q \). Here, for small wave steepness, we keep only the linear term \( U q' \). \{Note, if the zonal flow varied in \( y \), \( U(y) \), it would have vorticity and there would be another linear term, \( vQ_y \) where \( Q \) is the PV of the mean state, \( f - U_y \).\} The wave equation becomes

\[
\frac{\partial q}{\partial t} + U q' x + v'(Q_y + q'_y) = 0
\]

but we cross off the nonlinear terms with error of order \( U/(L/T) \), to give

\[
\nabla^2 \psi_t + U \nabla^2 \psi_x + L_D^2 (\psi_t + U_0 \psi_x) + \beta \psi_x = 0
\]

Substitute a wave of the form \( \psi = \text{Real} (\exp(ikx + ily - i\sigma_0 t)) \), to give

\[
-k^2 (-i\sigma + U_0 k) - L_D^{-2} (-i\sigma + U_0 k) + \beta i k = 0
\]

or

\[
\sigma - U_0 k = \frac{-\beta k}{k^2 + l^2 + L_D^{-2}}
\]

We can draw the curves of constant frequency on the \( k,l \) plane as before, though it is easier to let Matlab do it. Choose the special case \( \sigma = 0 \) of stationary (standing) waves of zero frequency and make the upper surface rigid, so that \( L_D \rightarrow \infty \). The dispersion relation is then

\[
k[U_0 = \frac{\beta}{k^2 + l^2}]
\]

There are two roots

\[
k = 0; \quad k^2 + l^2 = \frac{\beta}{U_0}
\]

where we have also for simplicity gone to the rigid lid, \( L_D \rightarrow \infty \). The locus of possible wavenumbers is now a circle centered on the origin, \( k=0, l=0 \).
Fig. 5.7 Wavenumber space diagram for possible stationary ($\sigma=0$) Rossby waves with a uniform eastward mean flow. The radius of the semicircle is ($\beta/U_0)^{1/2}$. Arrows indicate the direction and magnitude of the group velocity. The vertical axis ($k=0$) is a special set of waves that are nearly zonal currents, yet propagate as Rossby waves (they have finite group velocity even as their intrinsic frequency vanishes). Rhines, J.Atmos.Sci. 2007

Doing the group velocity construction to see which wavenumbers will appear in which directions from the source of waves, we can see that the wave crests will be circles too, just as if there were no anisotropy in the waves! It is a little tricky to get the direction of the group velocity right for each (k,l) choice of wave, but can be done by thinking of the group velocity as the sum of the mean flow ($U_0, 0$) and the intrinsic group velocity of Rossby waves without a mean flow. This means that all of the waves with a westward component of group velocity are shifted to the east and appear downstream. The result is that the arrows normal to the circle always point to the right. In the end, this amounts to repeating the same waves twice, so on the figure above we show just $1/2$ the wavenumber circle. The wave crests in (x,y) space are also semicircles rather than circles because they all propagate downstream, with a component to the east. The full solution (exact) for flow over a cylindrical mountain (McCartney 1976), below, shows the pattern of streamlines and wavecrests.

**Blocking the flow with Rossby waves (optional section).** There is one more contribution to the wave pattern: the root $k=0$. This would seem to be a non-wave, just a pattern of zonal currents with no variation in the x-direction. But if the limit $\sigma \to 0$ is taken for the Rossby wave propagation we find a finite, strong group velocity toward the west even as the frequency vanishes. This is an expression of ‘blocking’, a band of zonal flow that *propagates* like a wave. See Fig. 5.7. It is a strong feature if you put a mountain in the flow (as in McCartney’s figure, yet he missed this part of the solution), as in the GFD lab altimetry figure below. The same thing happens in stratified, non-rotating flow over a
mountain: if slow enough, the flow is blocked upstream by the mountain and internal gravity waves are involved in setting up this blocking region.

Fig. 5.8 Two fundamental barotropic Rossby wave patterns: above, in a fluid at rest with an initial prescribed eddy of stream function. The amplitude is so small that the ‘eddy’ does not rotate, but disperses into waves. This is a β-plane fluid, with x and y Cartesian coordinates. The boundary conditions are periodic, so the waves re-enter the fluid after exiting the box. Shown are the streamfunction at three times, the initial condition is at left.

Below, stationary Rossby waves generated by eastward (westerly) flow over a cylindrical mountain (McCartney, *J.Fluid Mech.* 1975). The semicircular wavecrests are evident. Notice that a high pressure anticyclone sits on the upstream side of the mountain while a low pressure cyclone sits in the lee...this is lee cyclogenesis, and the pressure drag on the mountain (∫ p dh/dx dx or ∫ p dh) is substantial. Here the upper surface is rigid, L_D => ∞. Note the visual impression in both the resting mean ocean (above) and the eastward mean flow (below) of ‘waves on a mean flow’. This optical illusion occurs in the top figures due to the strange anisotropic physics of the spherical, rotating Earth.
Fig. 5.9 Two views of Rossby waves generated at a single frequency by a ‘wavemaker’; upper color figure, the wavemaker is a glass cylinder pushed in and out of the fluid at its surface. This is a polar β-plane in which the depth h gives us a variation of mean PV = f/h. The ‘North Pole’ is at the center, and the colored rings were on circles of latitude (radius=constant) before the waves arrived. Short, energetic waves appear east of the forcing. The ‘PV elasticity’ is very evident in videos of the experiment. It both makes waves possible and also resists turbulent mixing. Like the famous ozone hole over Antarctica, the PV gradient prevents the orange polar fluid from being mixed to lower latitude. The lower figure shows the pressure field or free surface elevation for a similar experiment (actually, here there is a mountain and the flow is sloshed periodically back and forth over it, by periodically changing the rotation rate). We can see the long Rossby wave propagating westward from the wavemaker, and spiraling poleward. This does not look very much like the β-plane solution, Fig. 5.4c because it is wrapped into cylindrical geometry, but the main features are the same. See www.ocean.washington.edu/research/gfd or Rhines, Lindahl & Mendez, J. Fluid Mech. 2007; Rhines, J. Atmos. Sci 2007.
Fig. 5.10 GFD lab experiment with an eastward mean flow over a mountain (a spherical
cap mountain located at 2 o’clock in the figure. The downstream Rossby waves are evident in the pressure field, and in particle paths shown in the lower figure. Upstream of the mountain there is a region of blocked flow, which is stagnant…the result of a fast upstream propagating Rossby wave seen on Fig. 5.7. The topography of the fluid surface (the pressure field) shows that the zonal flow, initially a solid-body rotation about the Pole, is concentrated into jet streams both downstream and upstream of the small mountain. This focusing of the zonal circulation into jets is an aspect of Rossby wave interaction with the mean flow.

Fig. 5.11  The same as the stationary Rossby wave experiment in Fig. 5.10 but with the mean flow reversed, so that it is westward. A mountain in a westward (‘easterly’) wind does not generate stationary Rossby waves because their intrinsic westward propagation cannot be brought to zero by the mean zonal current. Instead we see stationary inertial waves (non-geostrophic, shorter, intrinsic frequency > f). A Taylor column with nearly motionless fluid sits over the mountain. Also a shear line appears round a latitude circle, and as in 5.10 there are hundreds of small ‘craters’ which are convection cells (miniature tornadoes, with cyclonic rotation. Optical altimetry was developed in the UW GFD lab to make these images, which show subtle surface elevations of 1 micron (10⁻⁶ m) or less.
Fig. 5.12. The vorticity argument for the restoring effect that makes Rossby waves: northward deflection of an eastward current (westerly wind) conserves \( f + \zeta \) hence with \( f \approx f_0 + \beta y \) larger, \( \zeta \) is smaller and in fact negative. This anticyclonic vorticity steers the flow back toward the Equator. Conversely for the equatorward deflection: smaller \( f \) means larger \( \zeta \) which is cyclonic, and steers the current poleward.

Rossby waves: parcel argument. In terms of forces, we can suggest an oscillator equation for a fluid parcel: the northward-deflected parcel feels an extra Coriolis force \( U_0 \beta \Delta Y \) to the south and this causes an acceleration \( (\Delta Y)_{tt} \) southward. Now that is as seen by the fluid parcel. For a stationary observer the temporal acceleration becomes \( kU_0 \) where \( k \) is the x-wavenumber (this is just the Doppler shift visible in the expression \( (\sigma - kU_0) \) for the Rossby wave dispersion relation). So the oscillator equation for the fluid parcel, 
\[
(\Delta Y)_{tt} + (U_0 \beta) \Delta Y = 0
\]
becomes
\[
(kU_0)^2 \Delta Y \sim U_0 \beta \Delta Y
\]
or
\[
k^2 = \beta/U_0
\]
which is in fact our dispersion relation for \( l = 0 \). If we want to go back to waves in a fluid at rest, \( U_0 = 0 \), replace \( kU_0 \) by \( \sigma \) and we find \( \sigma = -\beta/k \) as we have found from the real derivation.

Be warned however, that there are some subtleties here related to the pressure field. I hope to clear these up soon, but, suffice it to say, many research papers are either wrong or unclear about the ‘restoring force for Rossby waves’.

Observations of Rossby waves or modes close to Rossby waves are numerous. Here we simply show the first observation of oceanic Rossby waves in the MODE experiment of 1973 (Fig. 5.5) using H.T. Rossby’s (son of C.G. Rossby) deep ocean SOFAR floats which tracked fluid particles at 1500m depth, and the striking observations of Chelton et al. (GRL 2007, Science 1995) of ocean surface elevation from the NASA and ESA altimetry satellites. The Rossby float observations are close to the barotropic mode in properties and the Chelton surface pressure waves are close to a 1st baroclinic mode Rossby wave in properties…nearly a 1½ layer model in which the waves are non-dispersive and propagate due westward. They are in both cases nonlinear, with wave steepness between 1 and 5.
Fig. 5.13 Wintertime average circulation in the northern hemisphere at 300 HPa (solid curves, contour interval $10^7 \text{ m}^2\text{sec}^{-1}$) and barotropic PV, $\zeta + f$ (dashed curves, contour interval $1x10^{-5} \text{ sec}^{-1}$. (Lau, 1979 J.Atmos.Sci). If these were stationary Rossby waves in the equivalent of the barotropic mode, the two fields would coincide, and they are not too far out of line. But, important baroclinic effects are present, which relate to downward flux of zonal momentum and also to poleward flux of heat. What generates these waves? It is a combination of the major mountain ranges, the Tibetan Plateau in Asia and Rockies in N America, plus the thermal forcing by the oceans. Both effects are sort of ‘wavenumber 2’ in scale: two mountain massifs and two major oceans. The storm tracks in Atlantic and Pacific are controlled by these forcing effects: note how the mean winds flow northeastward in the Atlantic sector: this is an important conduit for moisture- and heat flux from lower latitudes toward the Arctic.
Fig. 5.14 From Jin & Hoskins, *J.Atmos.Sci.* 1995, Stationary Rossby waves generated by forcing in the western equatorial Pacific. 15 days after thermal forcing was switched on, A basic wintertime mean flow is present. **upper:** meridional wind contours in the upper troposphere. These are ‘equivalent’ barotropic waves, not quite simple barotropic but strongly resembling them. **lower:** Streamfunction in upper troposphere which shows tropical response invisible in the meridional wind, in the form of equatorial Kelvin and Rossby waves.
Interaction between Rossby waves and mean circulation. We said at the beginning that Rossby waves tend not to be very linear: their ‘steepness’ often approaches 1 or greater. This makes them very interactive with the mean circulation. They ‘break’. The PV field ceases to be a smooth gradient but instead develops closed contours associated with the waves. In Dudley Chelton’s satellite videos we have estimated that the ‘waves’ are actually carrying fluid westward, as if Rossby waves are themselves a part of the general circulation. Biological communities can develop (plankton blooms) in their upwelled waters.
The oceans are in fact turbulent. At scales of a few km and greater this tends to be 'geostrophic' turbulence. The eddies that dominate the spectrum of energy (the 'synoptic' scale \( \sim 10^3 \) km in the atmosphere, the 'mesoscale' \( \sim 10^2 \) km in the ocean) are quasi-geostrophic and hydrostatic. Early on, researchers thought this turbulence probably acted like more familiar 3-dimensional turbulence, in mixing momentum and tracers. But thanks to PV dynamics it is nothing like that. The general circulation of oceans and atmosphere can be concentrated into jet streams and boundary currents by seemingly random turbulence…because Rossby wave dynamics organizes its 'randomness'. This idea of waves interacting with mean flows is far beyond this review of basic Rossby waves.

PV 'thinking' has been a great help in this research, and while it is difficult for most of us, it seems worth the trouble. For a brief discussion of this see Baldwin et al. *Science*, 315, 26 January 2007, 467-468; for a long discussion of PV thinking and its uses see the review article by Hoskins et al. *Quarterly J. Royal Meteorological Soc.* 1985.