8.1 1 Rossby waves and circulation, V: Wind-driven circulation: gyres, boundary currents and spin-up from rest.

We have seen the basic solution for the steady Sverdrup-Stommel model of the wind-driven circulation. This was comprised of a very simple interior solution where the meridional velocity was simply proportional to the curl of (wind-stress divided by \( f \)). Because this is the Ekman pumping velocity we are tempted to make a simple geometrical explanation, indicated at the end of lecture 3 (p 3-13). Earth’s rotation endows the fluid with stiffness along the direction of the rotation axis. If Ekman pumping velocity, \( w_e \), is directed down on the fluid from above, it is rather like an upper boundary being pressed down on the fluid. Almost like stiff metal poles, aligned parallel with \( \Omega \), the fluid is forced southward into ‘deeper water’ (deeper when the ocean is measured parallel with \( \Omega \)). This thickness, \( d \), of the spherical ocean varies like \( \sin \varphi \) and hence its slope \( \partial d/\partial y \) is proportional to \( \cos \varphi/a \), giving \( (\cos \varphi \cdot v \cdot h)/a = w_e \sin \varphi \) or \( \beta vh/f = w_e \) as we wrote on p. 3-13. The Sverdrup solution (Eqn. 7.8) with the u-velocity calculated from v using approximate non-divergence, \( u_x + v_y = 0 \), is shown in Fig. 8.1 from Gill, Chapter 12.

![Diagram of Sverdrup's steady wind-driven circulation for the case of a sinusoidally varying eastward stress. The assumed stress is shown in the panel at the left, with values for the latitudes that approximately apply in practice. Near 30°, where the surface air pressure is a maximum, the Ekman pumping also has a maximum magnitude and is directed downward. The solution is shown by solid contours at the right. The response in practice is mainly baroclinic (in the sense that the largest currents are confined to the upper layers), so the contours may be interpreted as corresponding to either dynamic height or thermocline depth. The direction of the wind-driven currents is shown by the arrows. The results of numerical calculations for real ocean basins look very similar [see, e.g., Anderson et al. (1979)]. The dotted lines at the bottom left of the diagram are added to show how the beginnings of the western boundary current (see Section 12.6) relate to the Sverdrup solution.](image)
Fig. 8.1 The Sverdrup circulation for a wind-stress directed east and west, varying sinusoidally with latitude (Gill Ch. 12).

In the complete steady solution including the western boundary current, Eqn. 7.8, the Sverdrup solution feeds fluid to the boundary current, which must accept it and move it to a different latitude. Think of the overwhelming control of the Earth’s rotation on the fluid’s potential vorticity (PV); by actively pumping the fluid, the winds can force it southward across the ‘free pathways’, \( f = \text{constant} \) (latitude circles). If there were a region of positive wind-stress curl west of the negative wind-stress curl in Fig. 8.1, then the fluid could return smoothly northward. But there is not such a region. Friction forces must arise to change the PV of the fluid particle so that it can recirculate northward. The fluid ‘rubs on the bottom’ to change its PV, so that it can return north and recirculate round the gyre.

Notice that the kinetic energy of the circulation is dominated by boundary current, even though it is not directly forced by the wind. There are many aspects of this solution which differ from the real ocean, but its general character seems to be correct. Three aspects that are not found to be correct are: (i), the PV in this linear solution is everywhere close to \( f/H \); the relative vorticity does not change it greatly (whereas the oceanic PV is very much different from \( f/H \)); (ii), the wind blows across the boundary current, perpendicular to it, so that it does not directly feed energy into it via the term \( \mathbf{r} \cdot \mathbf{u} \) (whereas the winter winds blow along the path of the actual Gulf Stream), and (iii), here the interior solution determines the boundary current flow (whereas in some respects the boundary currents in the real ocean can determine the interior flow).

Although the model so far is quite idealized, the occurrence of western boundary currents is obvious in observations of the real ocean, Figs 8.3, 8.4.

*Time-development of the circulation.* We have seen in numerical simulations that Rossby waves set up the general circulation, if the wind is switched on with the ocean at rest. This may not sound realistic, yet the seasonal and interannual changes in the winds are very great, for example the wintertime windstress exceeding the summer wind-stress by a large factor in the high northern latitudes. The atmospheric circulation lags behind its forcing (the sun’s heating) by only a month or two, so that the winds begin to accelerate in September. The barotropic ocean circulation which we have been modeling here lags behind the forcing (the winds) by just a few weeks. To see this, consider an idealized wind-forcing and an ocean without boundaries, for which

\[
F = A \cos kx \cos ly
\]

switched on at time \( t = 0 \). This might be a east-west wind-stress varying like \( \cos kx \sin ly \).

The solution to

\[
\nabla^2 \eta_\lambda + R \nabla^2 \eta_\lambda - \lambda^{-2} \eta_\lambda + \beta \eta_\lambda = F
\]
for \( R/\beta L \ll 1 \) involves a particular, forced solution \( \eta_p \) (the Sverdrup circulation) plus a homogeneous, free, wave solution \( \eta_h \) for which the right-hand side vanishes. At \( t=0 \) the forcing begins to spin up the fluid through the \( \nabla^2 \eta_t \) term, and soon, wave dynamics comes into play through the other terms.

Thus

\[
\eta_p = -(k \beta)^{-1} A \sin kx \cos ly
\]

and \( \eta_h \) is a wave chosen to just cancel \( \eta_p \) at time \( t=0 \). A choice that works is

\[
\eta_h = (k \beta)^{-1} A \sin(kx - \sigma t) \exp(-Rt) \cos ly
\]

\[
\sigma = -\beta k / (k^2 + l^2 + \lambda^{-2})
\]

so that

\[
\eta = (k \beta)^{-1} A \cos ly [\sin kx - \sin(kx - \sigma t) \exp(-Rt)]
\]

whose time average is just the Sverdrup circulation, with a slowly decaying transient wave. The time required to set up the circulation from rest is just the time for the two \( \sin \) components to separate from one another, or a time \( 1/\sigma \sim 1/\beta L \) where \( L \) is the length scale defined by \( k \) and \( l \). If we imagine that \( L \) represents the width of the Atlantic Ocean (~5000 km), then the circulation spins up in about a day (formally violating our assumption that the time scale is much greater than a day; yet, the very long barotropic Rossby waves do have large wave-speeds). The length-scale\(^2\) dependence of Rossby wave-speed means that 100 km scale waves (600 km wavelength) propagate at about 0.2 m sec\(^{-1}\), whereas a basin-filling long wave with 1000 km scale propagates 100 times as fast.

The first effects of density stratification: the 1 \( \frac{1}{2} \) layer model. The layered density of the ocean and atmosphere do not change the fact that there is a Rossby wave that acts as if the fluid were homogeneous: the barotropic mode of a stratified ocean is just the same as the wave in a single-layer, constant density ocean. Yet there is also an infinity of Rossby wave modes with vertical shear in the horizontal current field, known as baroclinic modes. Remarkably, they have just the same vertical structure as internal gravity waves for the same profile of density. The wave equations are in many cases separable into the product of a vertical structure (a function of \( z \)) and a horizontal wave (a function of \( x,y,t \)). Let us begin to see the effect of stratification by developing what is called the 1 \( \frac{1}{2} \) layer model. We imagine an infinitely deep ocean with density \( \rho_0 \) on top of which lies a thin layer of density \( \rho_0 - \Delta \rho \). The interface between layers is the ‘thermocline’, which varies in depth between 100m and 1000m. Its displacement is \( \eta' \), and the free surface displacement is \( \eta \). The wave equation for motions in the upper layer is similar to that for a single layer with a free surface.

\[
\begin{align*}
\nabla^2 \eta'_t - \lambda'^{-2} \eta'_t + \beta \eta'_x &= F \\
\lambda' &= c'/f
\end{align*}
\]

(8.1)
except that now $c'$, the speed of long internal gravity waves without rotation, is very much slower than $c$, the speed of external surface gravity waves. We now have two Rossby deformation radii, one based on the full gravity, $g$, ($\lambda \sim 5000$ km) and one based on the reduced gravity, $g'$, experienced at the density interface ($\lambda' \sim 10$ to 50 km). This model is best derived as a two-layer model with mean depths $H_1$ and $H_2$, then letting $H_2/H_1 \to \infty$. It is described fully in Gill section 6.2. Internal wave modes in a two-layer model have equal and opposite horizontal volume fluxes in the two layers, so that in this limit the velocity and kinetic energy of the lower layer become negligibly small. The motion is concentrated in the thin upper layer. In this limit it turns out that

$$c^2 = g(H_1 + H_2)$$

or

$$c^2 = (g\Delta \rho / \rho_0)H_1$$

There are now two possible modes, the \textit{barotropic} or \textit{external} mode, with horizontal velocities the same in both layers, minimal vertical movement of the interface, $\eta'$, and the \textit{baroclinic} or \textit{internal} mode which is new. The free surface movement is very slight for the internal mode. Because $g$ and $g'$ differ by a factor of order $10^3$ the modes are well separated: internal mode Rossby waves move the free surface only slightly compared to large (often tens of m) displacement of the thermocline, $\eta_i$. To motivate the vertical structure of the baroclinic mode, consider the vortex stretching in the two layers, in response to some vertical motion of the thermocline, $\eta_i$. The ratio horizontal velocities in upper and lower layer will be proportional to the vortex stretching, hence will be of order $H_1/H_2$. For this reason we neglect the small horizontal velocity of the deep lower layer.

This vast difference in Rossby deformation radius means that internal mode Rossby waves with length scale greater than 50 km or so (wavelengths greater than 300 km) obey a simplified wave equation

$$-\lambda^{-2} \eta'_{,t} + \lambda \beta \eta'_{,x} = F$$

and are non-dispersive, and propagate in only one direction, due westward. It is a remarkable and simple statement applying to most of the dynamical scales of relevance to the general circulation. The free waves, with zero on the right-hand side, are of the form

$$\eta' = \exp(ikx + ily - i\sigma t)$$

$$\sigma / k = \beta \lambda^{-2}$$

but it is more instructive to write the general solution,
\[ \eta' = m(x + \beta \lambda^2 t) \]

for any function \( m(\ ). \) The free baroclinic Rossby, for \( H_1 = 100 \text{m} \) and \( \Delta \rho/\rho_0 = 2 \times 10^{-3} \) has a Rossby radius \( \lambda' = \sqrt{(g' H_1)/f} = 45 \text{ km} \) and a wave-speed \( \beta \lambda^2 \) of \( 4.6 \times 10^{-2} \cos \varphi \text{ m sec}^{-1} \) (using \( \beta = (2\Omega/a) \cos \varphi = 2.28 \times 10^{-11} \cos \varphi \text{ m}^{-1} \text{sec}^{-1} \)). This is typically 2 cm sec\(^{-1}\) in the subtropics, which is only 2 km per day! At this rate the spin-up of a 5000 km wide ocean takes just days for the barotropic mode, yet of order 10 years for the thermocline to adjust. Recall that the propagation speed for a 1000 km scale barotropic Rossby wave is of order 20 m sec\(^{-1}\) or 2000 km per day.

This is a good point to go back to so-called Chap. 14, roughly p.38-41 ‘A simplified view of long baroclinic (Rossby) waves’. This long-wave limit has such a simple equation that it implies a simple physics, which is described there.

To now make a complete solution for the spin-up of the circulation we need once again to include a model of frictional effects near the western boundary. The problem without friction cannot handle the western boundary condition. Yet, if we retain some north-south (cos ly) structure, we will have a reflected short Rossby wave, as in the numerical simulations illustrated in class.

\[ -(l^2 + \lambda^2 \beta) \eta' + \beta \eta' = 0 \]

This solution is plotted in Figure 8.2. Notice how in mid-ocean, the Ekman pumping simply pushes the thermocline downward uniformly with respect to \( x \). It is the \( \eta = 0 \) boundary condition at the eastern boundary that propagates at the long wave speed \( \beta \lambda^2 \) westward, reshaping the thermocline into a slope. At the arrival of this wave, the north-south velocity changes from barotropic (uniform with depth and very weak) to baroclinic (vanishing in the deep lower layer, and strong in the upper layer, with thermal wind shear at the thermocline density interface). The short, reflected wave is very slow and with any friction is rapidly dissipated. Even before adding friction we see nearly the shape of the ultimate steady circulation: southward flow confined to the upper layer, obeying Sverdrup balance, with a thin layer of special dynamics near the western boundary.

Notice how the north-south wavenumber \( l \) and \( \lambda' \) have the same effect on the wave equation: non-dispersive Rossby waves occur when \( k << 1 \) or when \( k << 1/\lambda' \). It is this non-dispersive, slowly propagating solution that dominates the Topex/Poseidon altimeter satellite images of the ocean surface (Fig. 5.5). Ironically we see the internal-mode Rossby wave by its small effect on the sea-surface elevation, \( \eta \), even though we neglected this displacement in the dynamics. The slowness of propagation suggests that neglected nonlinear terms may become important; yet, evidently the waves do exist and dominate the long-period movement of the sea-surface.
Fig. 8.2 Development of the thermocline displacement (or eastward velocity) as a function of x, for various times after the switch-on of wind-stress. This is the same pattern seen in the simple numerical model animated on my lap-top.