1. The angular momentum of a ring of fluid expanding outward will be constant in a non-viscous flow symmetrical about the vertical axis. Thus

\[ r v = c \]

where \( r \) = radius, \( v \) = absolute azimuthal velocity observed in a non-rotating frame, \( c \) = constant.

In the rotating frame (whose azimuthal velocity is \( \Omega r \)) this translates to azimuthal velocity in the rotating frame, \( v' \), given by

\[ v' = \Omega (-r + c/r). \]

If in this frame the velocity vanishes at \( r = a \), then

\[ v' = \Omega (-r + a^2/r) \]

The angular velocity of the fluid is just \( v'/r \). Note that the variable \( r \) here is following an expanding ring of fluid (it is a ‘Lagrangian’ coordinate of the fluid ring). But since the situation is steady, \( r \) can be thought of as the ordinary ‘Eulerian’ coordinate.

This is very fast swirl velocity, as in a bathtub vortex (though here the fluid is going outward). The Stommel-Aron-Faller source driven flows had a much smaller velocity due to the blockage of the flow by walls. A scale horizontal velocity in the interior (away from boundary currents) comes from the vertical vorticity equation

\[ \beta v = f w \]

giving \( v \sim f/\beta H \). The vertical velocity \( w \) is given by the source flow strength, distributed uniformly over the surface area of the fluid.

2. Flow of stratified fluid over seafloor mountains and valleys. This is the generalization of the unstratified Taylor column problem to density stratified fluid and finite flow velocity. The imposed barotropic (depth-independent) mean flow is \( U \) in the x-direction. Perturbation velocities \( (u, v, w) \) develop due to the topography.

The potential vorticity, for constant stratification \( N \), is

\[ q = \psi_{xx} + \psi_{yy} + f_0^2/N^2 \psi_{zz} + f_0 \]

is conserved, and we said it was uniform (the same everywhere), far upstream, so it remains constant everywhere, ignoring dissipation. Thus the perturbation PV equals zero,

\[ \psi_{xx} + \psi_{yy} + f_0^2/N^2 \psi_{zz} = 0 \]  \( (1) \)

The bottom lies at \( z = -h(x,y) \) where

\[ h = H + A \cos k_x x \cos l_y y \]

and the boundary condition at the bottom is that the flow is parallel with the boundary \( \vec{u} \cdot \vec{n} = 0 \) where \( \vec{n} \) is a unit vector normal to the boundary. This can be written

\[ w = -(u h_x + v h_y) \text{ at } z = -h(x,y) \]

Note that \( h \) is depth, not height of the topography, giving a minus sign. We linearize the boundary condition for small height mountains, and strong mean flow, so that \( u << U, \ v << U \). Thus we have

\[ w = -U h_x \text{ at } z = -H \]

From derivation of the PV equation (which started with \( \frac{D\zeta}{Dt} = f w \zeta \) being vertical vorticity) we recall that \( w \) can be written in terms of the geostrophic streamfunction \( \psi \) as

\[ w = -\frac{f_0}{N^2} \frac{D\psi_z}{Dt} \]

\[ \quad = -\frac{U f_0}{N^2} \psi_{xx} \]  \( (2) \)

for steady flow (\( \partial/\partial t = 0 \)).
Use separation of variables on the PV equation, looking for a solution of the form
\[ \psi = g_1(z) \ g_2(x,y) \]
The boundary condition suggests that
\[ g_2 = B\cos k_0x \cos l_0y. \]
Substitute in (1) to give
\[ (-k_0^2-l_0^2)g_1 + \frac{f_0}{N^2} \frac{d^2g_1}{dz^2} = 0 \]
with solutions \( g_1 \) proportional to \( \exp(\kappa(z+H)) \) where \( \kappa^2 = (N^2/f_0^2)(k_0^2 + l_0^2) \).
Plug into the boundary condition (2),
\[ \frac{d}{dx}[\frac{Uf_0}{N^2} \kappa B \cos(k_0x) \cos(l_0y)] = -U h_x = -UA \frac{d}{dx}[\cos(k_0x) \cos(l_0y)] \]
This confusion is simply saying \( \frac{f_0}{N^2} \psi_z = h - H, \)
or \[ B = -A \frac{(N^2/f_0\kappa)}{(-k_0^2 + l_0^2)^{1/2}}. \]
\(1/\kappa\) is the height to which the effect of the topography penetrates. It is interesting that \( f_0 \) does not appear explicitly in the amplitude of \( \psi \). The plot of the solution is on the website.

So \[ \psi = -A \frac{((k_0^2 + l_0^2)^{1/2} \exp(-\kappa(z+H)) \cos k_0x \cos l_0y; \]
\[ \kappa^2 = (N^2/f_0^2)(k_0^2 + l_0^2). \]
or simply \[ \psi = \frac{N}{(k_0^2 + l_0^2)^{1/2}} e^{-\kappa(z+H)} h(x,y) \]
plus the uniform mean flow which adds a streamfunction \( -Uy \).
This solution has the fluid following the topography at the bottom, so that \( \rho' \), the perturbation density is just \( h \frac{d\rho}{dz} \) (this means that the total density \( \rho + \rho' \) can be conserved following a fluid parcel along a streamline.

The solution is very interesting. The flow deflects left and right, so that the vorticity above the hills is negative, and above the valleys is positive (basic vortex stretching). But the deflections die away exponentially in \( z \), over the ‘Prandtl scale’ \( f_0L/N \) where \( L \) is the horizontal length scale \( (k_0^2 + l_0^2)^{-1/2} \). These are ‘Taylor cones’, in which the Taylor column effect is eventually suppressed by stable stratification.

In fact if we put a rigid lid on top with \( w = 0 \) at \( z = 0 \) as a second boundary condition, this gives
\[ \psi_z = 0 \quad \text{on} \quad z = 0. \]
For the vertical eigenfunction \( g_1(z) \) we now need both the two exponentials, \( \exp(kz) \) and \( \exp(-kz) \). It happens to go more quickly if we combine these into one function, \( \cosh(z) \), the hyperbolic cosine: \( \cosh(z) = \frac{1}{2} (e^z + e^{-z}) \) and hyperbolic sine \( \sinh(z) = \frac{1}{2} (e^z - e^{-z}) \). The reason for doing this is that \( \cosh(z) \) has zero slope at \( z=0 \), so this is just what we need for the upper boundary condition. Now our solution looks as before except the vertical velocity totally vanishes at the rigid surface \( z = 0 \). Try
\[ \psi = B \cosh(kz) \ h(x,y) \]
and now the lower boundary condition, which is \( \psi_z = N^2/f_0 \ h(x,y) \) at \( z = -H \), becomes
\[ Bk\sinh(kH) = \frac{N^2}{f_0} A \]
because the derivative of a \( \cosh \) is a \( \sinh \). Finally we have
\[ \psi = \frac{N^2 \cosh \kappa z}{f_0 \kappa \sinh \kappa H} h(x, y) \]

Now what’s really interesting is that if we make the horizontal scale \( L \) of the mountains >> the Rossby radius, our solution becomes barotropic, with nearly the same streamlines at all depths (since the \( \cosh(\kappa z) \) function becomes just a constant and \( \sinh(\kappa H) \sim \kappa H \)).

In this large-\( L \) limit the solution is

\[ \psi = \frac{f_0}{(k_0^2 + l_0^2) H} h(x, y) \]

But this is just the barotropic, unstratified result of conserving PV in a single fluid layer, where \( \nabla^2 \psi - f_0 h'/H = 0 \)

The stratified solution follows the basic scaling of rotating stratified GFD, that the ratio of horizontal length scales to vertical length scales of the flow is \( N/f_0 \), or, \( \frac{N H_1}{f L} \sim 1 \) where here the ‘\( H_1 \)’ is the z-scale of variation of the flow, not the total fluid depth…or, \( \frac{L}{\lambda} \sim 1 \) where \( \lambda \) is the Rossby radius based on the flow scale \( H_1 \). It is interesting that as \( \kappa H \ll 1 \) (the barotropic, small fluid depth, large \( L \) limit) the horizontal velocity develops a big barotropic component so that \( \psi_z/\psi \sim \kappa^2 H \) rather than \( \kappa \), which it was for the other extreme, deep fluid case.

Here the topographic contours are the ‘checkerboard’ pattern, the deep flow is the black streamlines and the upper level flow is the green streamlines. Note the veering of the direction of the horizontal velocity between different depths. This can be simply related to the vertical velocity field, \( w(x, y) \), through the thermal wind equation.

There is much more to say about flow over bumps. In particular, bumps create eddies and turbulent mixing, so that PV is no longer conserved. The drag force that bumps exert on fluids is considerable, yet in this simple theory it is zero.