GFD-2 Problem Set 4
out: 14 May 2010
back: 21 May 2010

This develops our course material in two directions: waves on a zonal current and energy considerations.

Please record your study group/homework group on your paper if you have one.

1. Rossby waves on an eastward current. This is a standard problem for atmospheric sciences. The dominant stationary waves of the northern hemisphere are modeled as Rossby waves forced by mountains, with the eastward mean winds.

The linearized equation for the streamfunction is

\[
\frac{Dq}{Dt} = 0; \quad q = \frac{f_0 + \beta y + \zeta}{h}
\]

\[
\Rightarrow U \frac{\partial}{\partial x} \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = 0
\]

where U is the constant (given) mean eastward velocity. So we are looking for steady solutions for waves that stand still, their westward phase speed being opposed by eastward advection. More generally we could retain the time-dependent and nonlinear terms, and frictional effects.

Following the course notes,

• write the dispersion relation for standing waves, \( \psi = \text{Real}(A \exp(ikx + ily)) \). If you include time-dependence, the Doppler shift will be evident in the dispersion relation, \( \sigma - Uk \) is the frequency seen by an observer moving at a speed U. The physics does transform this way, except for the fact that when you move at a different speed on a rotating (i.e. accelerating) Earth, some things do change (like the geopotential ‘horizons’).

• Complete the discussion in those notes by showing how the group velocity of the waves depends on the direction of propagation (hence showing that the waves appear downstream of a region of forcing (e.g., a mountain).

Here it is good to think of the group velocity as simply the sum of the usual group velocity for Rossby waves without a mean flow, plus the vector velocity \((U,0)\).

• Plot the group velocity on the \((k,l)\) curve showing waves for a constant \(U\), showing the magnitude and direction. Show that the wave field that develops after the flow is started (say, over a mountain) will propagate downstream at a speed \(2U\).

• Do likewise for the strange solution \(k=0\), which we have described in class: as you let the frequency of normal Rossby waves (with \(U=0\)) go to zero, you find a westward group velocity that is non-zero….describing nearly zonal, nearly steady currents which are established by Rossby wave propagation. You can see these waves in GFD lab publications on ‘jets’ (J.Atmos Sci 2007, J.Fluid Mech 2006…copies outside OSB 319)

We might add some bottom topography which would excite the waves…as it is we are not solving a complete problem since we don’t specify the source of the waves.
2. Stratified Taylor columns. We can exercise both thermal wind dynamics and PV dynamics now that we have an equation for stratified PV conservation. Solve for the steady flow over a hill and valley topography,

\[ h = H + A \cos(k_0x + l_0y) \]

...actually the choice \( h = H + A\cos(k_0x)\cos(l_0y) \) would be a little more interesting.

in a uniform eastward current, \( U \), on an f-plane (i.e. \( \beta = 0 \)). The stratification is uniform, \( N = \text{constant} \).

The equation is

\[ \frac{Dq}{Dt} = 0; \quad q = \nabla^2 \psi + \frac{f^2}{N^2} \psi \]

and we will simply say that somewhere upstream there is no topography and so PV is uniform. Thus solve \( q = 0 \) with boundary conditions

\[ \psi \rightarrow 0 \text{ as } z \rightarrow \infty \text{ and } \]
\[ \hat{u} \cdot \hat{n} = 0 \text{ at } z = -h(x,y). \]

The latter condition is approximately:

\[ w = U \partial h/\partial x \text{ at } z = -H, \text{ the mean depth.} \]

The trick now is to express \( w \) in terms of \( \psi \) and then solve by separation of variables. Make plots showing how the streamline pattern varies with depth, and compare with our Taylor column intuition. Show carefully how thermal wind equations express the change in horizontal velocity with depth and show how PV is conserved following a fluid parcel (how layer thickness and relative vorticity trade off). Notice how the direction of the velocity vector \((u,v)\) varies in the vertical (a plot known as a hodograph, of \( u \) and \( v \) at a single position \((x,y)\) is very interesting. It is a single curve with axes \( u \) and \( v \), and it often has the form of a part of a spiral.